

INFINITESIMALLY LIPSCHITZ FUNCTIONS ON METRIC SPACES

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ABSTRACT. For a metric space X , we study the space $D^\infty(X)$ of bounded functions on X whose infinitesimal Lipschitz constant is uniformly bounded. $D^\infty(X)$ is compared with the space $\text{LIP}^\infty(X)$ of bounded Lipschitz functions on X , in terms of different properties regarding the geometry of X . We also obtain a Banach-Stone theorem in this context. In the case of a metric measure space, we also compare $D^\infty(X)$ with the Newtonian-Sobolev space $N^{1,\infty}(X)$. In particular, if X supports a doubling measure and satisfies a local Poincaré inequality, we obtain that $D^\infty(X) = N^{1,\infty}(X)$.

1. INTRODUCTION

Recent years have seen many advances in geometry and analysis, where first order differential calculus has been extended to the setting of spaces with no a priori smooth structure; see for instance [Am, He1, He2, S]. The notion of derivative measures the infinitesimal oscillations of a function at a given point, and gives information concerning for instance monotonicity. In general metric spaces we do not have a derivative, even in the weak sense of Sobolev spaces. Nevertheless, if f is a real-valued function on a metric space (X, d) and x is a point in X , one can use similar measurements of sizes of first-order oscillations of f at small scales around x , such as

$$D_r f(x) = \frac{1}{r} \sup \left\{ |f(y) - f(x)| : y \in X, d(x, y) \leq r \right\}.$$

On one hand, this quantity does not contain as much information as standard derivatives on Euclidean spaces does (since we omit the signs) but, on the other hand, it makes sense in more general settings since we do not need any special behavior of the underlying space to define it. In fact, if we look at the superior limit of the above expression as r tends to 0 we almost recover in many cases, as in the Euclidean or Riemannian setting, the standard notion of derivative. More precisely, given a continuous function $f : X \rightarrow \mathbb{R}$, the *infinitesimal Lipschitz constant* at a point $x \in X$ is defined as follows:

$$\text{Lip } f(x) = \limsup_{r \rightarrow 0} D_r f(x) = \limsup_{\substack{y \rightarrow x \\ y \neq x}} \frac{|f(x) - f(y)|}{d(x, y)}.$$

Recently, this functional has played an important role in several contexts. We just mention here the construction of differentiable structures in the setting of metric measure spaces [Ch, K], the theory of upper gradients [HK, Sh2], or the Stepanov's differentiability theorem [BRZ].

This concept gives rise to a class of function spaces, *infinitesimally Lipschitz function spaces*, which contains in some sense infinitesimal information about the

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functions,

$$D(X) = \{f : X \longrightarrow \mathbb{R} : \|\text{Lip } f\|_\infty < +\infty\}.$$

This space $D(X)$ clearly contains the space $\text{LIP}(X)$ of Lipschitz function and a first approach should be comparing such spaces. In Corollary 2.6 we give sufficient conditions on the metric space X to guarantee the equality between $D(X)$ and $\text{LIP}(X)$. A powerful tool which transforms bounds on infinitesimal oscillation to bounds on maximal oscillation is a kind of mean value theorem (see Lemma 2.5 in [S]). In fact, the largest class of spaces for which we obtain a positive answer is the class of quasi-length spaces, which has a characterization in terms of such mean type value theorem. In particular, this class includes quasi-convex spaces. In addition, we present some examples for which $\text{LIP}(X) \neq D(X)$ (see Examples 2.7 and 2.8).

At this point, it seems natural to approach the problem of determining which kind of spaces can be classified by their infinitesimal Lipschitz structure. Our strategy will be to follow the proof in [GJ2] where the authors find a large class of metric spaces for which the algebra of bounded Lipschitz functions determines the Lipschitz structure for X . A crucial point in the proof is the use of the Banach space structure of $\text{LIP}(X)$. Thus, we endow $D(X)$ with a norm which arises naturally from the definition of the operator Lip . This norm is not complete in the general case, as it can be seen in Example 3.3. However, there is a wide class of spaces, the locally radially quasiconvex metric spaces (see Definition 3.1), for which $D^\infty(X)$ (bounded infinitesimally Lipschitz functions) admits the desired Banach space structure. Moreover, for such spaces, we obtain a kind of Banach-Stone theorem in this framework (see Theorem 4.7).

If we have a measure on the metric space, we can deal with many more problems. In this line, there are for example generalizations of classical Sobolev spaces to the setting of arbitrary metric measure spaces. It seems that Hajszl  was the first who introduced Sobolev type spaces in this context [Ha2]. He defined the spaces $M^{1,p}(X)$ for $1 \leq p \leq \infty$ in connection with maximal operators. It is well known that $M^{1,\infty}(X)$ is in fact the space of bounded Lipschitz functions on X . Shanmugalingam in [Sh2] introduced, using the notion of upper gradient (and more generally weak upper gradients) the Newtonian spaces $N^{1,p}(X)$ for $1 \leq p < \infty$. The generalization to the case $p = \infty$ is straightforward and we will compare the function spaces $D^\infty(X)$ and $\text{LIP}^\infty(X)$ with such Sobolev space, $N^{1,\infty}$. From Cheeger's work [Ch], metric spaces with a doubling measure and a Poincar  inequality admit a differentiable structure with which Lipschitz functions can be differentiated almost everywhere. Under the same hypotheses we prove in Corollary 5.16 the equality of all the mentioned spaces. Furthermore, if we just require a local Poincar  inequality we obtain $M^{1,\infty}(X) \subseteq D^\infty(X) = N^{1,\infty}(X)$. For further information about more generalizations of Sobolev spaces on metric measure spaces see [Ha1].

We organized the work as follows. In Section 2 we will introduce *infinitesimally Lipschitz function spaces* $D(X)$ and we look for conditions regarding the geometry of the metric spaces we are working with in order to understand in which cases the infinitesimal Lipschitz information yields the global Lipschitz behavior of a function. Moreover, we show the existence of metric spaces for which $\text{LIP}(X) \subsetneq D(X)$. In Section 3 we introduce the class of *locally radially quasiconvex metric spaces* and we prove that the space of bounded infinitesimally Lipschitz function can be endowed with a natural Banach space structure. The purpose of Section 4 is to state a kind of Banach-Stone theorem in this context while the aim of Section 5 is to compare

the function spaces $D^\infty(X)$ and $\text{LIP}^\infty(X)$ with Sobolev spaces in metric measure spaces.

2. INFINITESIMALLY LIPSCHITZ FUNCTIONS

Let (X, d) be a metric space. Given a function $f : X \rightarrow \mathbb{R}$, the *infinitesimal Lipschitz constant* of f at a non isolated point $x \in X$ is defined as follows:

$$\text{Lip } f(x) = \limsup_{\substack{y \rightarrow x \\ y \neq x}} \frac{|f(x) - f(y)|}{d(x, y)}.$$

If x is an isolated point we define $\text{Lip } f(x) = 0$. This value is also known as *upper scaled oscillation* (see [BRZ]) or as *pointwise infinitesimal Lipschitz number* (see [He2]).

Examples 2.1. (1) If $f \in C^1(\Omega)$ where Ω is an open subset of Euclidean space, or of a Riemannian manifold, then $\text{Lip } f = |\nabla f|$.

(2) Let \mathbb{H} be the first Heisenberg group, and consider an open subset $\Omega \subset \mathbb{H}$. If $f \in C_H^1(\Omega)$, that is, f is H -continuously differentiable in Ω , then $\text{Lip } f = |\nabla_H f|$ where $\nabla_H f$ denotes the horizontal gradient of f . For further details see [Ma].

(3) If (X, d, μ) is a metric measure space which admits a measurable differentiable structure $\{(X_\alpha, \mathbf{x}_\alpha)\}_\alpha$ and $f \in \text{LIP}(X)$, then $\text{Lip } f(x) = |d^\alpha f(x)|$ μ -a.e., where $d^\alpha f$ denotes the Cheeger's differential. For further information about measurable differentiable structures see [Ch, K].

Loosely speaking, the operator $\text{Lip } f$ estimates some kind of infinitesimal Lipschitzian property around each point. Our first aim is to see under which conditions a function $f : X \rightarrow \mathbb{R}$ is Lipschitz if and only if $\text{Lip } f$ is a bounded functional. It is clear that if f is a L -Lipschitz function, then $\text{Lip } f(x) \leq L$ for every $x \in X$. More precisely, we consider the following spaces of functions:

- ◇ $\text{LIP}(X) = \{f : X \rightarrow \mathbb{R} : f \text{ is Lipschitz}\}$
- ◇ $D(X) = \{f : X \rightarrow \mathbb{R} : \sup_{x \in X} \text{Lip } f(x) = \|\text{Lip } f\|_\infty < +\infty\}$.

We denote by $\text{LIP}^\infty(X)$ (respectively $D^\infty(X)$) the space of bounded Lipschitz functions (respectively, bounded functions which are in $D(X)$) and $\mathcal{C}(X)$ will denote the space of continuous functions on X . It is not difficult to see that for $f \in D(X)$, $\text{Lip } f$ is a Borel function on X and that $\|\text{Lip}(\cdot)\|_\infty$ yields a seminorm in $D(X)$. In what follows, $\|\cdot\|_\infty$ will denote the supremum norm whereas $\|\cdot\|_{L^\infty}$ will denote the essential supremum norm, provided we have a measure on X . In addition, $\text{LIP}(\cdot)$ will denote the Lipschitz constant.

Since functions with uniformly bounded infinitesimal Lipschitz constant have a flavour of differentiability it seems reasonable to determine if the infinitesimally Lipschitz functions are in fact continuous. Namely,

Lemma 2.2. *Let (X, d) be a metric space. Then $D(X) \subset \mathcal{C}(X)$.*

Proof. Let $x_0 \in X$ be a non isolated point and $f \in D(X)$. We are going to see that f is continuous at x_0 . Since $f \in D(X)$ we have that $\|\text{Lip } f\|_\infty = M < \infty$, in particular, $\text{Lip } f(x_0) \leq M$. By definition we have that

$$\text{Lip } f(x_0) = \inf_{r > 0} \sup_{\substack{d(x_0, y) \leq r \\ y \neq x_0}} \frac{|f(x_0) - f(y)|}{d(x_0, y)}.$$

Fix $\varepsilon > 0$. Then, there exists $r > 0$ such that

$$\frac{|f(x_0) - f(z)|}{d(x_0, z)} \leq \sup_{\substack{d(x_0, y) \leq r \\ y \neq x_0}} \frac{|f(x_0) - f(y)|}{d(x_0, y)} \leq M + \varepsilon \quad \forall z \in B(x_0, r),$$

and so

$$|f(x_0) - f(z)| \leq (M + \varepsilon)d(x_0, z) \quad \forall z \in B(x_0, r).$$

Thus, if $d(x_0, z) \rightarrow 0$ then $|f(x_0) - f(z)| \rightarrow 0$, and so f is continuous at x_0 . \square

Now we look for conditions regarding the geometry of the metric space X under which $\text{LIP}(X) = D(X)$ (respectively $\text{LIP}^\infty(X) = D^\infty(X)$). As it can be expected, we need some kind of *connectedness*. In fact, we are going to obtain a positive answer in the class of *length spaces* or, more generally, of *quasi-convex spaces*. Recall that the *length* of a continuous curve $\gamma : [a, b] \rightarrow X$ in a metric space (X, d) is defined as

$$\ell(\gamma) = \sup \left\{ \sum_{i=0}^{n-1} d(\gamma(t_i), \gamma(t_{i+1})) \right\}$$

where the supremum is taken over all partitions $a = t_0 < t_1 < \dots < t_n = b$ of the interval $[a, b]$. We will say that a curve γ is *rectifiable* if $\ell(\gamma) < \infty$. Now, (X, d) is said to be a *length space* if for each pair of points $x, y \in X$ the distance $d(x, y)$ coincides with the infimum of all lengths of curves in X connecting x with y . Another interesting class of metric spaces, which contains length spaces, are the so called *quasi-convex spaces*. Recall that a metric space (X, d) is *quasi-convex* if there exists a constant $C > 0$ such that for each pair of points $x, y \in X$, there exists a curve γ connecting x and y with $\ell(\gamma) \leq Cd(x, y)$. As one can expect, a metric space is *quasi-convex* if, and only if, it is bi-Lipschitz homeomorphic to some length space.

We begin our analysis with a technical result.

Lemma 2.3. *Let (X, d) be a metric space and let $f \in D(X)$. Let $x, y \in X$ and suppose that there exists a rectifiable curve $\gamma : [a, b] \rightarrow X$ connecting x and y , that is, $\gamma(a) = x$ and $\gamma(b) = y$. Then, $|f(x) - f(y)| \leq \|\text{Lip } f\|_\infty \ell(\gamma)$.*

Proof. Since $f \in D(X)$, we have that $M = \|\text{Lip } f\|_\infty < +\infty$. Fix $\varepsilon > 0$. For each $t \in [a, b]$ there exists $\rho_t > 0$ such that if $z \in B(\gamma(t), \rho_t) \setminus \{\gamma(t)\}$ then

$$|f(\gamma(t)) - f(z)| \leq (M + \varepsilon)d(\gamma(t), z).$$

Since γ is continuous, there exists $\delta_t > 0$ such that

$$I_t = (t - \delta_t, t + \delta_t) \subset \gamma^{-1}(B(\gamma(t), \rho_t)).$$

The family of intervals $\{I_t\}_{t \in [a, b]}$ is an open covering of $[a, b]$ and by compactness it admits a finite subcovering which will be denoted by $\{I_{t_i}\}_{i=0}^{n+1}$. We may assume, refining the subcovering if necessary, that an interval I_{t_i} is not contained in I_{t_j} for $i \neq j$. If we relabel the indices of the points t_i in non-decreasing order, we can now choose a point $p_{i, i+1} \in I_{t_i} \cap I_{t_{i+1}} \cap (t_i, t_{i+1})$ for each $1 \leq i \leq n-1$. Using the auxiliary points that we have just chosen, we deduce that:

$$d(x, \gamma(t_1)) + \sum_{i=1}^{n-1} \left[d(\gamma(t_i), \gamma(p_{i, i+1})) + d(\gamma(p_{i, i+1}), \gamma(t_{i+1})) \right] + d(\gamma(t_n), y) \leq \ell(\gamma),$$

and so $|f(x) - f(y)| \leq (M + \varepsilon)\ell(\gamma)$. Finally, since this is true for each $\varepsilon > 0$, we conclude that $|f(x) - f(y)| \leq \|\text{Lip } f\|_\infty \ell(\gamma)$, as wanted. \square

As a straightforward consequence of the previous result, we deduce

Corollary 2.4. *If (X, d) is a quasi-convex space then $\text{LIP}(X) = D(X)$.*

The proof of the previous result is based on the existence of curves connecting each pair of points in X and whose length can be estimated in terms of the distance between the points. A reasonable kind of spaces in which we can approach the problem of determining if $\text{LIP}(X)$ and $D(X)$ coincide, are the so called *chainable spaces*. It is an interesting class of metric spaces containing length spaces and quasi-convex spaces. Recall that a metric space (X, d) is said to be *well-chained* or *chainable* if for every pair of points $x, y \in X$ and for every $\varepsilon > 0$ there exists an ε -chain joining x and y , that is, a finite sequence of points $z_1 = x, z_2, \dots, z_\ell = y$ such that $d(z_i, z_{i+1}) < \varepsilon$, for $i = 1, 2, \dots, \ell - 1$. In such spaces there exist “chains” of points which connect two given points, and for which the distance between the *nodes*, which are the points z_1, z_2, \dots, z_ℓ , is arbitrary small. However, throughout some examples we will see that there exists chainable spaces for which the spaces of functions $\text{LIP}(X)$ and $D(X)$ do not coincide (see Example 2.7). Nevertheless, if we work with a metric space X in which we can control the number of nodes in the chain between two points in terms of the distance between that points, then we will obtain a positive answer to our problem. A chainable space for which there exists a constant K (which only depends on X) such that for every $\varepsilon > 0$ and for every $x, y \in X$ there exists an ε -chain $z_1 = x, z_2, \dots, z_\ell = y$ such that

$$(\ell - 1)\varepsilon \leq K(d(x, y) + \varepsilon)$$

is called a *quasi-length space*. In Lemma 2.5. [S], Semmes gave a characterization of quasi-length spaces in terms of a condition which reminds a kind of “mean value theorem”.

Lemma 2.5. *A metric space (X, d) is a quasi-length space if and only if there exists a constant K such that for each $\varepsilon > 0$ and each function $f : X \rightarrow \mathbb{R}$ we have that*

$$|f(x) - f(y)| \leq K(d(x, y) + \varepsilon) \sup_{z \in X} D_\varepsilon f(z)$$

for each $x, y \in X$, where

$$D_\varepsilon f(z) = \frac{1}{\varepsilon} \sup \left\{ |f(y) - f(z)| : y \in X, d(z, y) \leq \varepsilon \right\}.$$

The previous characterization allows us to give a positive answer to our problem for quasi-length spaces. More precisely, we have the following:

Corollary 2.6. *Let (X, d) be a quasi-length space. Then, $\text{LIP}(X) = D(X)$.*

Proof. We have to check that $D(X) \subset \text{LIP}(X)$. Let $f \in D(X)$ and denote by $M = \|\text{Lip } f\|_\infty < +\infty$. Since X is a quasi-length space we obtain, applying Lemma 2.5, that there exists a constant $K \geq 1$ such that

$$|f(x) - f(y)| \leq K(d(x, y) + \varepsilon) \sup_{z \in X} D_\varepsilon f(z)$$

for each $x, y \in X$ and each $\varepsilon > 0$. Thus, if we take the superior limit when ε tends to zero we deduce:

$$|f(x) - f(y)| \leq K d(x, y) \sup_{z \in X} \text{Lip } f(z) = K M d(x, y)$$

for each $x, y \in X$. Thus, f is a KM -Lipschitz function and we are done. \square

We will see in 3.5 that the converse of Corollary 2.6 is true under more restrictive hypothesis.

Next, let us see that there exist metric spaces for which $\text{LIP}(X) \subsetneq D(X)$. We will approach this by constructing two metric spaces for which $\text{LIP}^\infty(X) \neq D^\infty(X)$. In the first example we see that the equality fails “for large distances” while in the second one it fails “for infinitesimal distances”.

Example 2.7. Define $X = [0, \infty) = \bigcup_{n \geq 1} [n-1, n]$, and write $I_n = [n-1, n]$ for each $n \geq 1$. Consider the sequence of functions $f_n : [0, 1] \rightarrow \mathbb{R}$ given by

$$f_n(x) = \begin{cases} x & \text{if } x \in [0, \frac{1}{n}] \\ \frac{nx+n-1}{n^2} & \text{if } x \in [\frac{1}{n}, 1]. \end{cases}$$

For each pair of points $x, y \in I_n$, we write $d_n(x, y) = f_n(|x - y|)$, and we define a metric on X as follows. Given a pair of points $x, y \in X$ with $x < y$, $x \in I_n$, $y \in I_m$ we define

$$d(x, y) = \begin{cases} d_n(x, y) & \text{if } n = m \\ d_n(x, n) + \sum_{i=n+1}^{m-1} d_i(i-1, i) + d_m(m-1, y) & \text{if } n < m \end{cases}$$

A straightforward computation shows that d is in fact a metric and it coincides locally with the Euclidean metric d_e . More precisely,

$$\text{if } x \in I_n, \text{ on } J^x = (x - \frac{1}{n+1}, x + \frac{1}{n+1}) \text{ we have that } d|_{J^x} = d_e|_{J^x}.$$

Next, consider the bounded function $g : X \rightarrow \mathbb{R}$ given by

$$g(x) = \begin{cases} 2k - x & \text{if } x \in I_{2k}, \\ x - 2k & \text{if } x \in I_{2k+1}. \end{cases}$$

Let us check that $g \in D^\infty(X) \setminus \text{LIP}^\infty(X)$. Indeed, let $x \in X$ and assume that there exists $n \geq 1$ such that $x \in I_n$. Then, we have that if $y \in J^x$,

$$\text{Lip } f(x) = \limsup_{\substack{y \rightarrow x \\ y \neq x}} \frac{|g(x) - g(y)|}{d(x, y)} = \limsup_{\substack{y \rightarrow x \\ y \neq x}} \frac{|x - y|}{|x - y|} = 1.$$

Therefore, $g \in D^\infty(X)$.

On the other hand, for each positive integer n we have $|g(n-1) - g(n)| = 1$ and $d(n-1, n) = f_n(1) = \frac{2n-1}{n^2}$. Thus, we obtain that

$$\lim_{n \rightarrow \infty} \frac{|g(n-1) - g(n)|}{d(n-1, n)} = \lim_{n \rightarrow \infty} \frac{1}{\frac{2n-1}{n^2}} = \infty$$

and so g is not a Lipschitz function.

In particular, since $\text{LIP}(X) \neq D(X)$, we deduce by Corollary 2.6 that X is not a quasi-convex space. However, it can be checked that X is a chainable space. \square

Example 2.8. Consider the set

$$X = \{(x, y) \in \mathbb{R}^2 : y^3 = x^2, -1 \leq x \leq 1\} = \{(t^3, t^2), -1 \leq t \leq 1\},$$

and let d be the restriction to X of the Euclidean metric of \mathbb{R}^2 . We define the bounded function

$$g : X \rightarrow \mathbb{R}, (x, y) \mapsto g(x, y) = \begin{cases} y & \text{if } x \geq 0, \\ -y & \text{if } x \leq 0. \end{cases}$$

Let us see that $g \in D^\infty(X) \setminus \text{LIP}^\infty(X)$.

Indeed, if $t \neq 0$, it can be checked that $\text{Lip } g(t^3, t^2) \leq 1$. On the other hand, at the origin we have

$$\text{Lip } g(0, 0) = \limsup_{(x, y) \rightarrow (0, 0)} \frac{|g(x, y) - g(0, 0)|}{d((x, y), (0, 0))} = \limsup_{t \rightarrow 0} \frac{t^2}{\sqrt{(t^3)^2 + (t^2)^2}} = 1.$$

Thus, we obtain that $\|\text{Lip } f\|_\infty = 1$ and so $g \in D^\infty(X)$. Take now two symmetric points from the cusp with respect to the y -axis, that is, $A_t = (t^3, t^2)$ and $B_t = (-t^3, t^2)$ for $0 < t < 1$. In this case, we get $d(A_t, B_t) = 2t^3$ and $|f(A_t) - f(B_t)| = t^2 - (-t^2) = 2t^2$. If t tends to 0, we have

$$\lim_{t \rightarrow 0^+} \frac{|f(A_t) - f(B_t)|}{d(A_t, B_t)} = \lim_{t \rightarrow 0^+} \frac{2t^2}{2t^3} = \lim_{t \rightarrow 0^+} \frac{1}{t} = +\infty.$$

Thus, g is not a Lipschitz function. \square

In general, if X is non compact space we have that

$$\text{LIP}(X) \subsetneq \text{LIP}_{\text{loc}}(X) \cap D(X) \begin{array}{c} \hookrightarrow \text{LIP}_{\text{loc}}(X) \\ \hookleftarrow D(X) \end{array} \begin{array}{c} \hookleftarrow \mathcal{C}(X) \\ \hookrightarrow \end{array}$$

where $\text{LIP}_{\text{loc}}(X)$ denotes the space of locally Lipschitz functions. Recall that in 2.7 we have constructed a function $f \in \text{LIP}_{\text{loc}}(X) \cap D(X) \setminus \text{LIP}(X)$. In addition, there is no inclusion relation between $\text{LIP}_{\text{loc}}(X)$ and $D(X)$. Indeed, consider for instance the metric space $X = \bigcup_{i=1}^{\infty} B_i \subset \mathbb{R}$ with the Euclidean distance where $B_i = B(i, 1/3)$ denotes the open ball centered at $(i, 0)$ and radius $1/3$. One can check that the function $f(x) = ix$ if $x \in B_i$ is locally Lipschitz whereas $f \notin D(X)$ because $\|\text{Lip } f\|_\infty = \infty$. On the other hand, the function g in Example 2.8 belongs to $D(X) \setminus \text{LIP}_{\text{loc}}(X)$.

3. A BANACH SPACE STRUCTURE FOR INFINITESIMALLY LIPSCHITZ FUNCTIONS

In this section we search for sufficient conditions to have a converse for Corollary 2.6. We begin introducing a kind of metric spaces which will play a central role throughout this section. In addition, for such spaces, we will endow the space of functions $D^\infty(X)$ and $D(X)$ with a Banach structure.

Definition 3.1. Let (X, d) be a metric space. We say that X is *locally radially quasi-convex* if for each $x \in X$, there exists a neighborhood U^x and a constant $K_x > 0$ such that for each $y \in U^x$ there exists a rectifiable curve α in U^x connecting x and y such that $\ell(\gamma) \leq K_x d(x, y)$.

Note that the spaces introduced in the Examples 2.7 and 2.8 are locally radially quasi-convex. Observe that there exist locally radially quasi-convex spaces which are not locally quasi-convex. Indeed, let $X = \bigcup_{n=1}^{\infty} \{(x, \frac{x}{n}) : x \in \mathbb{R}\}$ and d be the restriction to X of the Euclidean metric of \mathbb{R}^2 . It can be checked that (X, d) is locally radially quasi-convex but it is not locally quasi-convex.

Next, we endow the space $D^\infty(X)$ with the following norm:

$$\|f\|_{D^\infty} = \max\{\|f\|_\infty, \|\text{Lip } f\|_\infty\}$$

for each $f \in D^\infty(X)$.

Theorem 3.2. Let (X, d) be a locally radially quasi-convex metric space. Then, $(D^\infty(X), \|\cdot\|_{D^\infty})$ is a Banach space.

Proof. Let $\{f_n\}_n$ be a Cauchy sequence in $(D^\infty(X), \|\cdot\|_{D^\infty})$. Since $\{f_n\}_n$ is uniformly Cauchy, there exists $f \in \mathcal{C}(X)$ such that $f_n \rightarrow f$ with the norm $\|\cdot\|_\infty$. Let us see that $f \in D(X)$ and that $\{f_n\}_n$ converges to f with respect to the seminorm $\|\text{Lip}(\cdot)\|_\infty$.

Indeed, let $x \in X$. Since (X, d) is locally radially quasi-convex, there exist a neighborhood U^x and a constant $K_x > 0$ such that for each $y \in U^x$ there exists a rectifiable curve γ which connects x and y such that $\ell(\gamma) \leq K_x d(x, y)$. By Lemma 2.3, we find that for each $y \in U^x$ and for each $n, m \geq 1$

$$|f_n(x) - f_m(x) - (f_n(y) - f_m(y))| \leq \|\text{Lip}(f_n - f_m)\|_\infty K_x d(x, y).$$

Let $r > 0$ be such that $B(x, r) \subset U_x$ and let $y \in B(x, r)$. We have that

$$\begin{aligned} \left| \frac{f_n(x) - f_m(x)}{r} - \frac{f_n(y) - f_m(y)}{r} \right| &\leq \|\text{Lip}(f_n - f_m)\|_\infty K_x \frac{d(x, y)}{r} \\ &\leq \|\text{Lip}(f_n - f_m)\|_\infty K_x. \end{aligned}$$

Let $\varepsilon > 0$. Since $\{f_n\}_n$ is a Cauchy sequence with respect to the seminorm $\|\text{Lip}(\cdot)\|_\infty$, there exists $n_1 \geq 1$ such that if $n, m \geq n_1$, then

$$\|\text{Lip}(f_n - f_m)\|_\infty < \frac{\varepsilon}{4K_x}.$$

Thus, for each $r > 0$ such that $B(x, r) \subset U_x$ and for each $n, m \geq n_1$, we have the following chain of inequalities

$$\begin{aligned} \left| \frac{f_n(x) - f_n(y)}{r} - \frac{f_m(x) - f_m(y)}{r} \right| &\leq \left| \frac{f_n(x) - f_m(x)}{r} - \frac{f_n(y) - f_m(y)}{r} \right| \leq \|\text{Lip}(f_n - f_m)\|_\infty K_x < \frac{\varepsilon}{4} \end{aligned}$$

for each $y \in B(x, r)$.

In particular, for each $n \geq n_1$, we obtain that

$$\begin{aligned} \left| \frac{f_n(x) - f_n(y)}{r} \right| &\leq \left| \frac{f_n(x) - f_n(y)}{r} - \frac{f_{n_1}(x) - f_{n_1}(y)}{r} \right| \\ &\quad + \frac{|f_{n_1}(x) - f_{n_1}(y)|}{r} < \frac{|f_{n_1}(x) - f_{n_1}(y)|}{r} + \frac{\varepsilon}{4}. \end{aligned}$$

Thus, the previous inequality implies, upon taking the supremum over $B(x, r)$, that

$$\sup_{y \in B(x, r)} \left\{ \frac{|f_n(x) - f_n(y)|}{r} \right\} \leq \sup_{y \in B(x, r)} \left\{ \frac{|f_{n_1}(x) - f_{n_1}(y)|}{r} \right\} + \frac{\varepsilon}{4}$$

for each $r > 0$ such that $B(x, r) \subset U_x$.

On the other hand, for $\text{Lip}(f_{n_1})(x)$, there exists $r_0 > 0$, such that if $0 < r < r_0$, then $B(x, r) \subset U_x$ and

$$\sup_{y \in B(x, r)} \left\{ \frac{|f_{n_1}(x) - f_{n_1}(y)|}{r} \right\} \leq \text{Lip}(f_{n_1})(x) + \frac{\varepsilon}{4}.$$

Hence, for each $n \geq n_1$ and each $0 < r < r_0$, we obtain that

$$\sup_{y \in B(x, r)} \left\{ \frac{|f_n(x) - f_n(y)|}{r} \right\} \leq \text{Lip}(f_{n_1})(x) + \frac{2\varepsilon}{4}.$$

Since f_n is a Cauchy sequence with respect to the seminorm $\|\text{Lip}(\cdot)\|_\infty$, then the sequence of real numbers $\|\text{Lip}(f_n)\|_\infty$ is a Cauchy sequence too and so there

exists $M > 0$ such that $\|\text{Lip}(f_n)\|_\infty < M$ for each $n \geq 1$. In particular, for each $n \geq n_1$ and $0 < r < r_0$, we obtain the following:

$$\sup_{x \in B(x, r)} \left\{ \frac{|f_n(x) - f_n(y)|}{r} \right\} < \text{Lip}(f_{n_1})(x) + \frac{2\varepsilon}{4} \leq \|\text{Lip}(f_{n_1})\|_\infty + \frac{\varepsilon}{2} \leq M + \frac{\varepsilon}{2}.$$

Now, let us see what happens with f . If $n \geq n_1$, $0 < r < r_0$ and $y \in B(x, r)$, we have that

$$\begin{aligned} \frac{|f(x) - f(y)|}{r} &\leq \frac{|f(x) - f_n(x)|}{r} + \frac{|f_n(x) - f_n(y)|}{r} + \frac{|f_n(y) - f(y)|}{r} \\ &\leq \frac{|f(x) - f_n(x)|}{r} + \frac{|f_n(y) - f(y)|}{r} + M + \frac{\varepsilon}{2}. \end{aligned}$$

Since $\{f_n\}_n$ converges uniformly to f , it converges pointwise to f and so there exists $n \geq n_1$ such that

$$|f(x) - f_n(x)| + |f_n(y) - f(y)| < \frac{\varepsilon r}{2}.$$

Putting all above together we deduce that

$$\frac{|f(x) - f(y)|}{r} \leq M + \varepsilon.$$

Thus, that inequality implies, upon taking the infimum over $B(x, r)$ and letting r tending to 0 that

$$\text{Lip}(f)(x) \leq M + \varepsilon$$

for each $x \in X$. Now, if $\varepsilon \rightarrow 0$, we have that $\text{Lip}(f)(x) \leq M$ for each $x \in X$. And so $\|\text{Lip } f\|_\infty \leq M < +\infty$ which implies $f \in D(X)$.

To finish the proof, let us see that $\|\text{Lip}(f_n - f)\|_\infty \rightarrow 0$. Using the above notation we have that if $n, m \geq n_1$ and $0 < r < r_0$

$$\begin{aligned} \frac{|f_n(x) - f(x) - (f_n(y) - f(y))|}{r} &\leq \frac{|f_n(x) - f_m(x) - (f_n(y) - f_m(y))|}{r} \\ &\quad + \frac{|f_m(x) - f(x)|}{r} + \frac{|f(y) - f_m(y)|}{r} \leq \frac{|f_m(x) - f(x)|}{r} + \frac{|f(y) - f_m(y)|}{r} + \frac{\varepsilon}{4}. \end{aligned}$$

The sequence $\{f_n\}_n$ converges uniformly to f and, in particular, it converges pointwise to f . Thus, there exists $n \geq n_1$ such that

$$|f(x) - f_n(x)| + |f_n(y) - f(y)| < \frac{\varepsilon r}{2}.$$

Hence, we have

$$\frac{|f_n(x) - f(x) - (f_n(y) - f(y))|}{r} < \varepsilon.$$

Thus, we deduce that if $n \geq n_1$, then $\text{Lip}(f_n - f)(x) \leq \varepsilon$. This is true for each $x \in X$, and so we obtain that $\|\text{Lip}(f_n - f)\|_\infty \leq \varepsilon$ if $n \geq n_1$. Therefore, we have that $\|\text{Lip}(f_n - f)\|_\infty \rightarrow 0$. Thus, we conclude that $(D^\infty(X), \|\cdot\|_{D^\infty})$ is a Banach space as wanted. \square

Let us see however that in general $(D^\infty(X), \|\cdot\|_{D^\infty})$ is not a Banach space.

Example 3.3. Consider the connected metric space $X = X_0 \cup \bigcup_{n=1}^\infty X_n \cup G \subset \mathbb{R}^2$ with the metric induced by the Euclidean one, where $X_0 = \{0\} \times [0, +\infty)$, $X_n = \{\frac{1}{n}\} \times [0, n]$, $n \in \mathbb{N}$ and $G = \{(x, \frac{1}{x}) : 0 < x \leq 1\}$. For each $n \in \mathbb{N}$ consider the sequence of functions $f_n : X \rightarrow [0, 1]$ given by

$$f_n\left(\frac{1}{k}, y\right) = \begin{cases} \frac{k-y}{k\sqrt{k}} & \text{if } 1 \leq k \leq n \\ 0 & \text{if } k > n, \end{cases}$$

and $f_n(x, y) = 0$ if $x \neq \frac{1}{k} \forall k \in \mathbb{N}$. Observe that $f_n(\frac{1}{k}, 0) = \frac{1}{\sqrt{k}}$ and $f_n(\frac{1}{k}, k) = 0$ if $1 \leq k \leq n$. Since $\text{Lip } f_n(\frac{1}{k}, y) = \frac{1}{k\sqrt{k}}$ and $\text{Lip } f_n(x, y) = 0$ if $x \neq \frac{1}{k} \forall k \in \mathbb{N}$, we have that $f_n \in D^\infty(X)$ for each $n \geq 1$. In addition, if $1 < n < m$,

$$\|f_n - f_m\|_\infty = \frac{1}{\sqrt{n+1}} \quad \text{and} \quad \|\text{Lip}(f_n - f_m)\|_\infty = \frac{1}{(n+1)\sqrt{n+1}}.$$

Thus, we deduce that $\{f_n\}_n$ is a Cauchy sequence in $(D^\infty(X), \|\cdot\|_{D^\infty})$. However, if $f_n \rightarrow f$ in D^∞ then $f_n \rightarrow f$ pointwise. Then $f_m(\frac{1}{n}, 0) = \frac{1}{\sqrt{n}}$ for each $m \geq n$ and so $f(\frac{1}{n}, 0) = \frac{1}{\sqrt{n}}$ and $f(0, 0) = 0$. Thus, we obtain that

$$\text{Lip}(f)(0, 0) \geq \lim_{n \rightarrow \infty} \frac{|f((\frac{1}{n}), 0) - f(0, 0)|}{d(\frac{1}{n}, 0)} = \lim_{n \rightarrow \infty} \frac{\frac{1}{\sqrt{n}}}{\frac{1}{n}} = +\infty,$$

and so $f \notin D^\infty(X)$. This means that $(D^\infty(X), \|\cdot\|_{D^\infty})$ is not a Banach space.

Theorem 3.4. *Let (X, d) be a connected locally radially quasi-convex metric space and let $x_0 \in X$. If we consider on $D(X)$ the norm $\|\cdot\|_D = \max\{|f(x_0)|, \|\text{Lip}(\cdot)\|_\infty\}$, then $(D(X), \|\cdot\|_D)$ is a Banach space.*

Proof. By hypothesis, for each $y \in X$, there exists a neighborhood U^y such that for each $z \in U^y$, there exists a rectifiable curve in U^y connecting z and y . Since X is connected, there exists a finite sequence of points y_1, \dots, y_m such that $U^{y_k} \cap U^{y_{k+1}} \neq \emptyset$ for $k = 1, \dots, m-1$, $x \in U^{y_1}$ and $x_0 \in U^{y_m}$. Now, for each $k = 1 \dots m$, choose a point $z_k \in U^{y_k} \cap U^{y_{k+1}}$. To simplify notation we write $z_0 = x_0$ and $z_{m+1} = x$. For each $k = 1 \dots m$, we choose a curve γ_k which connects z_k with z_{k+1} . Taking $\gamma = \gamma_0 \cup \dots \cup \gamma_m$ we obtain a rectifiable curve γ which connects x_0 and x .

Let us see now that $(D(X), \|\cdot\|_D)$ is a Banach space. Indeed, let $\{f_n\}_n$ be a Cauchy sequence. We consider the case on which $f_n(x_0) = 0$ for each $n \geq 1$. The general case can be done in a similar way. By combining the previous argument with Lemma 2.3, we obtain that for $n, m \geq 1$ and for each $x \in X$, we have that

$$|f_n(x) - f_m(x)| \leq \|\text{Lip}(f_n - f_m)\|_\infty \ell(\gamma)$$

where γ is a rectifiable curve connecting x and x_0 . Since $\{f_n\}_n$ is a Cauchy sequence with respect to the seminorm $\|\text{Lip}(\cdot)\|_\infty$, the sequence $\{f_n(x)\}_n$ is a Cauchy sequence for each $x \in X$, and therefore, it converges to a point $y = f(x)$. Then, in particular, $\{f_n\}_n$ converges pointwise to a function $f : X \rightarrow \mathbb{R}$.

Next, one finds using the same strategy as in Theorem 3.2 (where we have just used the pointwise convergence) that a Cauchy sequence $\{f_n\}_n \subset D(X)$ such that $f_n(x_0) = 0$ for each $n \geq 1$, converges in $(D(X), \|\cdot\|_D)$ to a function $f \in D(X)$. \square

We are now prepared to state the converse of Corollary 2.6.

Corollary 3.5. *Let (X, d) be a connected locally radially quasi-convex metric space such that $\text{LIP}(X) = D(X)$. Then X is a quasi-length space.*

Proof. In view of Lemma 2.5 we have to prove that there exists $K > 0$ such that for each $\varepsilon > 0$ and each function $f : X \rightarrow \mathbb{R}$ we have that:

$$|f(x) - f(y)| \leq K(d(x, y) + \varepsilon) \sup_{z \in X} D_\varepsilon f(z) \quad \forall x, y \in X \quad (*).$$

Indeed, let $\varepsilon > 0$. If $\sup_{z \in X} D_\varepsilon f(z) = \infty$, then $(*)$ is trivially true. Thus, we may assume that $\sup_{z \in X} D_\varepsilon f(z) < \infty$. Since $\|\text{Lip } f\|_\infty \leq \sup_{z \in X} D_\varepsilon f(z)$ then $f \in D(X)$ and we distinguish two cases:

- (1) If $\|\text{Lip } f\|_\infty = 0$ then f is locally constant and so constant because X is connected. Therefore, the inequality trivially holds.
- (2) If $\|\text{Lip } f\|_\infty \neq 0$, using that $f \in D(X) = \text{LIP}(X)$, we have the following inequality

$$|f(x) - f(y)| \leq \text{LIP}(f)d(x, y) \quad \forall x, y \in X.$$

Now, fix a point $x_0 \in X$. Since $\text{LIP}(X) = D(X)$ is a Banach space with both norms

$$\|f\|_{\text{LIP}} = \max\{\text{LIP}(f), |f(x_0)|\} \quad \text{and} \quad \|f\|_D = \max\{\|\text{Lip } f\|_\infty, |f(x_0)|\},$$

(see Theorem 3.4 and e.g. [W]) and $\|\cdot\|_D \leq \|\cdot\|_{\text{LIP}}$, then there exists a constant $K > 0$ such that $\|\cdot\|_{\text{LIP}} \leq K\|\cdot\|_D$. Thus, if we consider the function $g = f - f(x_0)$ we have that

$$\text{LIP}(f) = \text{LIP}(g) = \|g\|_{\text{LIP}} \leq K\|g\|_D = K\|\text{Lip } g\|_\infty = K\|\text{Lip } f\|_\infty \quad (\heartsuit).$$

Thus, we obtain that

$$\begin{aligned} |f(x) - f(y)| &\leq \text{LIP}(f)d(x, y) \leq \text{LIP}(f)(d(x, y) + \varepsilon) \stackrel{(\heartsuit)}{\leq} K\|\text{Lip } f\|_\infty(d(x, y) + \varepsilon) \\ &\leq K \sup_{z \in X} D_\varepsilon f(z)(d(x, y) + \varepsilon) \quad \forall x, y \in X, \end{aligned}$$

as wanted. □

4. A BANACH-STONE THEOREM FOR INFINITESIMALLY LIPSCHITZ FUNCTIONS

There exist many results in the literature relating the topological structure of a topological space X with the algebraic or topological-algebraic structures of certain function spaces defined on it. The classical Banach-Stone theorem asserts that for a compact space X , the linear metric structure of $\mathcal{C}(X)$ endowed with the sup-norm determines the topology of X . Results along this line for spaces of Lipschitz functions have been recently obtained in [GJ2, GJ3]. In this section we prove two versions of the Banach-Stone theorem for the function spaces $D^\infty(X)$ and $D(X)$ respectively, where X is a locally radially quasi-convex space. Since in general $D(X)$ has not an algebra structure we will consider on it its natural unital vector lattice structure. On the other hand, on $D^\infty(X)$ we will consider both, its algebra and its unital vector lattice structures.

The concept of real-valued infinitesimally Lipschitz function can be generalized in a natural way when the target space is a metric space.

Definition 4.1. Let (X, d_X) and (Y, d_Y) be metric spaces. Given a function $f : X \rightarrow Y$ we define

$$\text{Lip } f(x) = \limsup_{\substack{y \rightarrow x \\ y \neq x}} \frac{d_Y(f(x), f(y))}{d_X(x, y)}$$

for each non-isolated $x \in X$. If x is an isolated point we define $\text{Lip } f(x) = 0$. We consider the following space of functions

$$D(X, Y) = \{f : X \rightarrow Y : \|\text{Lip } f\|_\infty < +\infty\}.$$

As we have seen in Lemma 2.2 we may observe that if $f \in D(X, Y)$ then f is continuous. It can be easily checked that we have also a Leibniz's rule in this context, that is, if $f, g \in D^\infty(X)$, then $\|\text{Lip}(f \cdot g)\|_\infty \leq \|\text{Lip } f\|_\infty \|g\|_\infty + \|\text{Lip } g\|_\infty \|f\|_\infty$. In this way, we can always endow the space $D^\infty(X)$ with a natural algebra structure. Note that $D^\infty(X)$ is *uniformly separating* in the sense that for every pair of

subsets A and B of X with $d(A, B) > 0$, there exists some $f \in D^\infty(X)$ such that $\overline{f(A)} \cap \overline{f(B)} = \emptyset$. In our case, if A and B are subsets of X with $d(A, B) = \alpha > 0$, then the function $f = \inf\{d(\cdot, A), \alpha\} \in \text{LIP}^\infty(X) \subset D^\infty(X)$ satisfies that $f = 0$ on A and $f = \alpha$ on B . In addition, we can endow either $D^\infty(X)$ or $D(X)$ with a natural unital vector lattice structure.

We denote by $\mathcal{H}(D^\infty(X))$ the set of all nonzero algebra homomorphisms $\varphi : D^\infty(X) \rightarrow \mathbb{R}$, that is, the set of all nonzero multiplicative linear functionals on $D^\infty(X)$. Note that in particular every algebra homomorphism $\varphi \in \mathcal{H}(D^\infty(X))$ is positive, that is, $\varphi(f) \geq 0$ when $f \geq 0$. Indeed, if f and $1/f$ are in $D^\infty(X)$, then $\varphi(f \cdot (1/f)) = 1$ implies that $\varphi(f) \neq 0$ and $\varphi(1/f) = 1/\varphi(f)$. Thus, if we assume that φ is not positive, then there exists $f \geq 0$ with $\varphi(f) < 0$. The function $g = f - \varphi(f) \geq -\varphi(f) > 0$, satisfies $g \in D^\infty(X)$, $1/g \in D^\infty(X)$ and $\varphi(g) = 0$ which is a contradiction.

Now, we endow $\mathcal{H}(D^\infty(X))$ with the topology of pointwise convergence (that is, considered as a topological subspace of $\mathbb{R}^{D^\infty(X)}$ with the product topology). This construction is standard (see for instance [I]), but we give some details for completeness. It is easy to check that $\mathcal{H}(D^\infty(X))$ is closed in $\mathbb{R}^{D^\infty(X)}$ and therefore is a compact space. In addition, since $D^\infty(X)$ separates points and closed sets, X can be embedded as a topological subspace of $\mathcal{H}(D^\infty(X))$ identifying each $x \in X$ with the point evaluation homomorphism δ_x given by $\delta_x(f) = f(x)$, for every $f \in D^\infty(X)$. We are going to see that X is dense in $\mathcal{H}(D^\infty(X))$. Indeed, given $\varphi \in \mathcal{H}(D^\infty(X))$, $f_1, \dots, f_n \in D^\infty(X)$, and $\varepsilon > 0$, there exists some $x \in X$ such that $|\delta_x(f_i) - \varphi(f_i)| < \varepsilon$, for $i = 1, \dots, n$. Otherwise, the function $g = \sum_{i=1}^n |f_i - \varphi(f_i)| \in D^\infty(X)$ would satisfy $g \geq \varepsilon$ and $\varphi(g) = 0$, and this is impossible since φ is positive. It follows that $\mathcal{H}(D^\infty(X))$ is a compactification of X . Moreover, every $f \in D^\infty(X)$ admits a continuous extension to $\mathcal{H}(D^\infty(X))$, namely by defining $\widehat{f}(\varphi) = \varphi(f)$ for all $\varphi \in \mathcal{H}(D^\infty(X))$.

Lemma 4.2. *Let (X, d) be a metric space and $\varphi \in \mathcal{H}(D^\infty(X))$. Then, $\varphi : D^\infty(X) \rightarrow \mathbb{R}$ is a continuous map.*

Proof. Let $f \in D^\infty(X)$. We know that it admits a continuous extension $\widehat{f} : \mathcal{H}(D^\infty(X)) \rightarrow \mathbb{R}$ so that $\widehat{f}(\varphi) = \varphi(f)$. Thus, since X is dense in $\mathcal{H}(D^\infty(X))$,

$$|\varphi(f)| = |\widehat{f}(\varphi)| \leq \sup_{\eta \in \mathcal{H}(D^\infty(X))} |\widehat{f}(\eta)| = \sup_{x \in X} |f(x)| \leq \|f\|_{D^\infty}$$

and we are done. \square

Recall that we have shown in Theorem 3.2 that if X is a locally radially quasi-convex space then $(D^\infty(X), \|\cdot\|_{D^\infty})$ is a Banach space. Using this in a crucial way, we next give some results which will give rise to a Banach-Stone theorem for $D^\infty(X)$.

Lemma 4.3. *Let (X, d_X) and (Y, d_Y) be locally radially quasi-convex metric spaces. Then, every unital algebra homomorphism $T : D^\infty(X) \rightarrow D^\infty(Y)$ is continuous for the respective D^∞ -norms.*

Proof. In order to prove the continuity of the linear map T , we apply the closed graph theorem. It is enough to check that given a sequence $\{f_n\}_n \subset D^\infty(X)$ with $\|f_n - f\|_{D^\infty}$ convergent to zero and $g \in D^\infty(X)$ such that $\|T(f_n) - g\|_{D^\infty}$ also convergent to zero, then $T(f) = g$. Indeed, let $y \in Y$, and let $\delta_y \in \mathcal{H}(D^\infty(Y))$ be the homomorphism given by the evaluation at y , that is, $\delta_y(h) = h(y)$. By Lemma

4.2, we have that $\delta_y \circ T \in \mathcal{H}(D^\infty(X))$ is continuous and so

$$T(f_n)(y) = (\delta_y \circ T)(f_n) \rightarrow (\delta_y \circ T)(f) = T(f)(y)$$

when $n \rightarrow \infty$.

On the other hand, since convergence in D^∞ -norm implies pointwise convergence, then $T(f_n)(y)$ converges to $g(y)$. That is, $T(f)(y) = g(y)$, for each $y \in Y$. Hence, $T(f) = g$ as wanted. \square

As a consequence, we obtain the following result concerning the composition of infinitesimally Lipschitz functions.

Proposition 4.4. *Let (X, d_X) and (Y, d_Y) be locally radially quasi-convex metric spaces and let $h : X \rightarrow Y$. Suppose that $f \circ h \in D^\infty(X)$ for each $f \in D^\infty(Y)$. Then $h \in D(X, Y)$.*

Proof. We begin by checking that h is a continuous function, that is, $h^{-1}(C)$ is closed in X for each closed subset C in Y . Let C be a closed subset of Y and $x_0 \in Y \setminus C$. Take $f = \inf\{d(\cdot, C), d(x_0, C)\} \in D^\infty(Y)$ which satisfies that $f(x_0) = 1$ and $f(y) = 0$ for each $y \in C$. Let us observe that $f(x) = 0$ if and only if $x \in C$ and so $f^{-1}(f(C)) = C$. Thus, since $f \circ h$ is continuous and $f(C) = 0$ is closed in \mathbb{R} , $h^{-1}(C) = h^{-1}(f^{-1}(f(C))) = (f \circ h)^{-1}(f(C))$ is closed in X .

Now, let $x_0 \in X$ such that $h(x_0)$ is not an isolated point. Note that if all points belonging to $h(X)$ are isolated we have that $\text{Lip } h(x) = 0$ for each $x \in X$ and so $\|\text{Lip } h\|_\infty = 0$, which implies that $h \in D(X, Y)$. Thus, we may assume that there exists $x_0 \in X$ such that $h(x_0)$ is not an isolated point. Let $f_{x_0} = \min\{d_Y(\cdot, h(x_0)), 1\} \in D^\infty(Y)$, since it is a Lipschitz function. We have that

$$\begin{aligned} \text{Lip}(f_{x_0} \circ h)(x_0) &= \limsup_{\substack{y \rightarrow x_0 \\ y \neq x_0}} \frac{|f_{x_0} \circ h(y) - f_{x_0} \circ h(x_0)|}{d_X(x_0, y)} = \limsup_{\substack{y \rightarrow x_0 \\ y \neq x_0}} \frac{|f_{x_0} \circ h(y)|}{d_X(x_0, y)} \\ &= \limsup_{\substack{y \rightarrow x_0 \\ y \neq x_0}} \frac{|\min\{d_Y(h(y), h(x_0)), 1\}|}{d_X(x_0, y)} \stackrel{(*)}{=} \text{Lip } h(x_0). \end{aligned}$$

The equality $(*)$ holds because, as we have checked above, the map h is continuous. Thus, we obtain that

$$\begin{aligned} \text{Lip } h(x_0) &= \text{Lip}(f_{x_0} \circ h)(x_0) \leq \|\text{Lip}(f_{x_0} \circ h)\|_\infty \leq \|f_{x_0} \circ h\|_{D^\infty(X)} \\ &\stackrel{(\heartsuit)}{\leq} K \|f_{x_0}\|_{D^\infty(Y)} \stackrel{(*)}{=} K \|\text{Lip}(f_{x_0})\|_\infty \quad (\dagger) \end{aligned}$$

for a certain constant $K > 0$ depending only on g . For (\heartsuit) we have used that, by Lemma 4.3, the homomorphism $T : D^\infty(Y) \rightarrow D^\infty(X)$, $g \rightarrow g \circ h$ is continuous. The inequality $(*)$ holds true because $\|f_x\|_{D^\infty(Y)} = \max\{\|f_{x_0}\|_\infty, \|\text{Lip}(f_{x_0})\|_\infty\}$, $\|f_{x_0}\|_\infty \leq 1$ and $\|\text{Lip}(f_{x_0})\|_\infty = 1$. It remains to check that, $\|\text{Lip}(f_{x_0})\|_\infty = 1$. Indeed,

$$\text{Lip } f_{x_0}(z) = \limsup_{\substack{z' \rightarrow z \\ z' \neq z}} \frac{|f_{x_0}(z') - f_{x_0}(z)|}{d_Y(z', z)}.$$

We have to distinguish three different cases:

- (i) If $d_Y(z, h(x_0)) > 1$, there exists a neighborhood V_z where $d_Y(z', h(x_0)) > 1$ for each $z' \in V_z$ and $f_{x_0}|_{V_z} = 1$. Thus, $\text{Lip } f_{x_0}(z) = 0$.

- (ii) If $d_Y(z, h(x_0)) < 1$, there exists a neighborhood V_z where $d_Y(z', h(x_0)) > 1$ for each $z' \in V_z$ and so

$$\text{Lip } f_{x_0}(z) = \limsup_{\substack{z' \rightarrow z \\ z' \neq z}} \frac{|d_Y(z', h(x_0)) - d_Y(z, h(x_0))|}{d_Y(z', z)} \leq \limsup_{\substack{z' \rightarrow z \\ z' \neq z}} \frac{d_Y(z', z)}{d_Y(z', z)} = 1.$$

- (iii) If $d_Y(z, h(x_0)) = 1$, then

$$\text{Lip } f_{x_0}(z) = \limsup_{\substack{z' \rightarrow z \\ z' \neq z}} \frac{1 - \min\{d_Y(z', h(x_0)), 1\}}{d_Y(z', z)}.$$

If $d_Y(z', h(x_0)) \geq 1$, then $1 - \min\{d_Y(z', h(x_0)), 1\} = 0$. On the other hand, if $d_Y(z', h(x_0)) < 1$, then

$$1 - \min\{d_Y(z', h(x_0)), 1\} = d_Y(z, h(x_0)) - d_Y(z', h(x_0)) \leq d_Y(z, z').$$

Hence, we deduce that $\text{Lip } f_{x_0}(z) \leq 1$.

On the other hand, since $h(x_0)$ is not an isolated point, $\text{Lip } f_{x_0}(z) = 1$ and so $\|\text{Lip } f\|_\infty = 1$ because we have seen that $\|\text{Lip } f\|_\infty \leq 1$. Then, upon taking the supremum over X in both sides of the inequality (†) we conclude that $\|\text{Lip } h\|_\infty \leq K$, as wanted. \square

Remark 4.5. If we look at Theorem 3.12 in [GJ2], where an analogous result to Proposition 4.4 for Lipschitz functions is obtained, we can see that the argument there is based on the fact that the distance can be expressed in terms of Lipschitz functions. In our case, we cannot use the same strategy since we do not know how to compare the values of an infinitesimally Lipschitz function at two arbitrary points of the space.

Finally, we need the following useful Lemma, which shows that the points in X can be topologically distinguished into $\mathcal{H}(D^\infty(X))$. It is essentially known (see for instance [GJ1]) but we give a proof for completeness.

Lemma 4.6. *Let (X, d) be a complete metric space and let $\varphi \in \mathcal{H}(D^\infty(Y))$. Then φ has a countable neighborhood basis in $\mathcal{H}(D^\infty(X))$ if, and only if, $\varphi \in X$.*

Proof. Suppose first that $\varphi \in \mathcal{H}(D^\infty(Y)) \setminus X$ has a countable neighborhood basis. Since X is dense in $\mathcal{H}(D^\infty(X))$, there exists a sequence (x_n) in X converging to φ . The completeness of X implies that (x_n) has no d -Cauchy sequence, and therefore there exist $\varepsilon > 0$ and a subsequence (x_{n_k}) such that $d(x_{n_k}, x_{n_j}) \geq \varepsilon$ for $k \neq j$. Now, the sets $A = \{x_{n_k} : k \text{ even}\}$ and $B = \{x_{n_k} : k \text{ odd}\}$ satisfy $d(A, B) \geq \varepsilon$, and since $D^\infty(X)$ is uniformly separating, there is a function $f \in D^\infty(X)$ with $\overline{f(A)} \cap \overline{f(B)} = \emptyset$. But this is a contradiction since f extends continuously to $\mathcal{H}(D^\infty(X))$ and φ is in the closure of both A and B .

Conversely, if $\varphi \in X$, consider B_n the open ball in X with center φ and radius $1/n$. Then the family $\{\overline{B_n}\}_n$ of the closures of B_n in $\mathcal{H}(D^\infty(X))$ is easily seen to be a countable neighborhood basis as required. \square

Now, we are in a position to show that the algebra structure of $D^\infty(X)$ determines the infinitesimal Lipschitz structure of a complete locally radially quasi-convex metric space. We say that two metric spaces X and Y are *infinitesimally Lipschitz homeomorphic* if there exists a bijection $h : X \rightarrow Y$ such that $h \in D(X, Y)$ and $h^{-1} \in D(Y, X)$.

Theorem 4.7. (Banach-Stone type) *Let (X, d_X) and (Y, d_Y) be complete locally radially quasi-convex metric spaces. The following are equivalent:*

- (a) X is infinitesimally Lipschitz homeomorphic to Y .
- (b) $D^\infty(X)$ is isomorphic to $D^\infty(Y)$ as unital algebras.
- (c) $D^\infty(X)$ is isomorphic to $D^\infty(Y)$ as unital vector lattices.

Proof. (a) \implies (b) If $h : X \rightarrow Y$ is an infinitesimally Lipschitz homeomorphism, then it is easy to check the map $T : D^\infty(Y) \rightarrow D^\infty(X)$, $f \mapsto T(f) = f \circ h$, is an isomorphism of unital algebras.

(b) \implies (a) Let $T : D^\infty(X) \rightarrow D^\infty(Y)$ be an isomorphism of unital algebras. We define $h : \mathcal{H}(D^\infty(Y)) \rightarrow \mathcal{H}(D^\infty(X))$, $\varphi \mapsto h(\varphi) = \varphi \circ T$. Let us see first that h is a homeomorphism. To reach that aim, it is enough to prove that h is bijective, closed and continuous. Since T is an isomorphism, $h^{-1}(\psi) = \psi \circ T^{-1}$ exists for every $\psi \in \mathcal{H}(D^\infty(X))$, and so h is bijective. In addition, once we check that h is continuous we will also have that h is closed because $\mathcal{H}(D^\infty(Y))$ is compact and $\mathcal{H}(D^\infty(X))$ is a Hausdorff space. Now consider the following diagram:

$$\begin{array}{ccccccc}
 Y & \hookrightarrow & \mathcal{H}(D^\infty(Y)) & \xrightarrow{h} & \mathcal{H}(D^\infty(X)) & \hookleftarrow & X \\
 \downarrow T(f) & & \downarrow \widehat{T(f)} & \searrow \widehat{f \circ h} & \downarrow \widehat{f} & & \downarrow f \\
 \mathbb{R} & & \mathbb{R} & & \mathbb{R} & & \mathbb{R}
 \end{array}$$

Here, \widehat{f} (respectively $\widehat{T(f)}$) denotes the continuous extension of f (respectively $T(f)$) to $\mathcal{H}(D^\infty(X))$. Thus, h is continuous if and only if $\widehat{f \circ h}$ is continuous for all $f \in D^\infty(X)$. Hence, it is enough to prove that $\widehat{f \circ h} = \widehat{T(f)}$. Since X is dense in $\mathcal{H}(D^\infty(X))$, it suffices to check that

$$\widehat{T(f)}(\delta_x) = \widehat{f \circ h}(\delta_x),$$

where δ_x denotes the evaluation homomorphism for each $x \in X$. It is clear that,

$$\widehat{f \circ h}(\delta_x) = (h \circ \delta_x)(f) = (\delta_x \circ T)(f) = \delta_{T(f)(x)} = \delta_x(Tf) = \widehat{T(f)}(\delta_x),$$

and so h is continuous.

By Lemma 4.6 we have that a point $\varphi \in \mathcal{H}(D^\infty(X))$ has a countable neighborhood basis in $\mathcal{H}(D^\infty(X))$ if and only if it corresponds to a point of X . Since the same holds for Y and $\mathcal{H}(D^\infty(Y))$ we conclude that $h(Y) = X$ and by Proposition 4.4 we have that $h|_Y \in D(Y, X)$. Analogously, $h^{-1}|_X \in D(X, Y)$ and so X and Y are infinitesimally Lipschitz homeomorphic.

To prove (b) \iff (c) We use that $D^\infty(X)$ is *closed under bounded inversion* which means that if $f \in D^\infty(X)$ and $f \geq 1$, then $1/f \in D^\infty(X)$. Indeed, if $f \in D^\infty(X)$ and $f \geq 1$, given $\varepsilon > 0$ there exists $r > 0$ such that

$$\frac{|f(x) - f(y)|}{d(x, y)} \leq \sup_{\substack{d(x, y) \leq r \\ y \neq x}} \frac{|f(x) - f(y)|}{d(x, y)} \leq M + \varepsilon \quad \forall y \in B(x, r) \quad (\star).$$

Thus, given $x \in X$,

$$\left| \frac{1}{f(y)} - \frac{1}{f(x)} \right| = \frac{|f(x) - f(y)|}{|f(x)f(y)|} \stackrel{(*)}{\leq} d(x, y)(M + \varepsilon) \quad \forall y \in B(x, r),$$

where inequality $(*)$ is obtained after applying (\star) and the fact that $|f(x)f(y)| \geq 1$. Thus, the conclusion follows from Lemma 2.3 in [GJ2]. \square

Corollary 4.8. *Let (X, d_X) and (Y, d_Y) be complete locally radially quasi-convex metric spaces. The following assertions are equivalent:*

- (a) X is infinitesimally Lipschitz homeomorphic to Y .

(b) $D(X)$ is isomorphic to $D(Y)$ as unital vector lattices.

Proof. (a) \implies (b) If $h : X \rightarrow Y$ is an infinitesimally Lipschitz homeomorphism, then it is clear that the map $T : D(Y) \rightarrow D(X)$, $f \mapsto T(f) = f \circ h$, is an isomorphism of unital vector lattices.

(b) \implies (a) It follows from Theorem 4.7, since each homomorphism of unital vector lattices $T : D(Y) \rightarrow D(X)$ takes bounded functions to bounded functions. Indeed, if $|f| \leq M$ then $|T(f)| = T(|f|) \leq T(M) = M$. \square

Next we deal with what we call *infinitesimal isometries* between metric spaces, related to infinitesimally Lipschitz functions.

Definition 4.9. Let (X, d_X) and (Y, d_Y) be metric spaces. We say that X and Y are *infinitesimally isometric* if there exists a bijection $h : X \rightarrow Y$ such that $\| \text{Lip } h \|_\infty = \| \text{Lip } h^{-1} \|_\infty = 1$.

Remark 4.10. We deduce from the proofs of Proposition 4.4 and Theorem 4.7 that two complete locally radially quasi-convex metric spaces X and Y are infinitesimally isometric if, and only if, there exists an algebra isomorphism $T : D^\infty(Y) \rightarrow D^\infty(X)$ which is an isometry for the $\| \cdot \|_{D^\infty}$ -norms (that is, $\|T\| = \|T^{-1}\| = 1$).

It is clear that if two metric spaces are locally isometric, then they are infinitesimally isometric. The converse is not true, as we can see throughout the following example.

Example 4.11. Let (X, d) be the metric space introduced in Example 2.8 and let (Y, d') be the metric space defined in the following way. Consider the interval $Y = [-1, 1]$ and let us define a metric on it as follows:

$$d'(t, s) = \begin{cases} d((t^3, t^2), (s^3, s^2)) & \text{if } t, s \in [-1, 0], \\ d((t^3, t^2), (s^3, s^2)) & \text{if } t, s \in [0, 1], \\ d((t^3, t^2), (0, 0)) + d((0, 0), (s^3, s^2)) & \text{if } t \in [-1, 0], s \in [0, 1]. \end{cases}$$

It is easy to see that d' defines a metric. We define

$$h : X \rightarrow Y, (t^3, t^2) \rightarrow t.$$

Let us observe that $\| \text{Lip } h \|_\infty = \| \text{Lip } h^{-1} \|_\infty = 1$ and so X and Y are infinitesimally isometric. However, at the origin $(0, 0)$, for each $r > 0$ we have that

$$d(z, y) \neq d'(h(z), h(y)) \quad \forall z, y \in B((0, 0), r).$$

Thus, h is an infinitesimal isometry, but not a local isometry. In fact, it can be checked that there is no local isometry $f : X \rightarrow Y$.

(4.12) Non complete case. If X is a metric space and \tilde{X} denotes its completion, then both metric spaces have the same uniformly continuous functions. Therefore, $\text{LIP}(X) = \text{LIP}(\tilde{X})$, and completeness of spaces cannot be avoided in the Lipschitzian case. We are interested in how completeness assumption works for the D -case. It would be useful to analyze if there exists a Banach-Stone theorem for not complete metric spaces.

Example 4.13. Let (X, d) be the metric space given by

$$X = \{(x, y) \in \mathbb{R}^2 : y^3 = x^2, -1 \leq x \leq 1\} = \{(t^3, t^2), -1 \leq t \leq 1\},$$

where d is the restriction to X of the Euclidean metric of \mathbb{R}^2 . Let (Y, d') be the metric space given by $Y = X \setminus \{0\}$ and $d' = d|_Y$. Observe that (X, d) is the completion of (Y, d') . The function

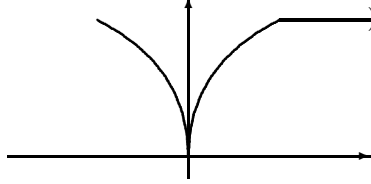
$$h : Y \rightarrow \mathbb{R}, (x, y) \mapsto \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x > 0, \end{cases}$$

belongs to $D(Y)$ but h cannot be even continuously extended to X . Thus, $D(Y) \neq D(X)$.

In the following example we construct a metric space X such that $D(X) = D(\tilde{X})$, where \tilde{X} denotes the completion of X , and so that X is not homeomorphic to \tilde{X} . This fact illustrates that, a priori, one cannot expect a conclusive result for the non complete case.

Example 4.14. Let X be a metric space defined as follows:

$$X = \{(t^3, t^2), -1 \leq t \leq 1\} \cup \{(x, 1) \in \mathbb{R}^2 : 1 \leq x < 2\} = A \cup B.$$



Now, we consider the completion of X :

$$\tilde{X} = \{(t^3, t^2), -1 \leq t \leq 1\} \cup \{(x, 1) \in \mathbb{R}^2 : 1 \leq x \leq 2\} = \tilde{A} \cup \tilde{B}.$$

Let $f \in D(X)$. First of all, $D(B) = \text{LIP}(B)$, since B is a quasi-length space, and so, by McShane's theorem (see [He1]), there exists $F \in \text{LIP}(\tilde{B})$ such that $F|_B = f$. Thus,

$$G(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in A = \tilde{A} \\ F(x, y) & \text{if } (x, y) \in \tilde{B}, \end{cases}$$

is a D -extension of f to the completion \tilde{X} . And so $D(X) = D(\tilde{X})$. However, X is not homeomorphic to \tilde{X} since \tilde{X} is compact but X is not.

5. SOBOLEV SPACES ON METRIC MEASURE SPACES

Along this section, we always assume that (X, d, μ) is a metric measure space, where μ is a *Borel regular measure*, that is, μ is an outer measure on a metric space (X, d) such that all Borel sets are μ -measurable and for each set $A \subset X$ there exists a Borel set B such that $A \subset B$ and $\mu(A) = \mu(B)$.

Our aim in this section is to compare the function spaces $D^\infty(X)$ and $\text{LIP}^\infty(X)$ with certain Sobolev spaces on metric-measure spaces. There are several possible extensions of the classical theory of Sobolev spaces to the setting of metric spaces equipped with a Borel measure. Following [Am] and [Ha1] we record the definition of $M^{1,p}$ spaces:

(5.1) Hajlasz-Sobolev space. For $0 < p \leq \infty$ the space $\widetilde{M}^{1,p}(X, d, \mu)$ is defined as the set of all functions $f \in L^p(X)$ for which there exists a function $0 \leq g \in L^p(X)$ such that

$$|f(x) - f(y)| \leq d(x, y)(g(x) + g(y)) \quad \mu - a.e. \quad (*).$$

As usual, we get the space $M^{1,p}(X, d, \mu)$ after identifying any two functions $u, v \in \widetilde{M}^{1,p}(X, d, \mu)$ such that $u = v$ almost everywhere with respect to μ . The space $M^{1,p}(X, d, \mu)$ is equipped with the norm

$$\|f\|_{M^{1,p}} = \|f\|_{L^p} + \inf_g \|g\|_{L^p},$$

where the infimum is taken over all functions $0 \leq g \in L^p(X)$ that satisfy the requirement (*).

In particular, if $p = \infty$ it can be shown that $M^{1,\infty}(X, d, \mu)$ coincides with $\text{LIP}^\infty(X)$ provided that $\mu(B) > 0$ for every open ball $B \subset X$ (see [Am]) and that $1/2 \|\cdot\|_{\text{LIP}^\infty} \leq \|\cdot\|_{M^{1,\infty}} \leq \|\cdot\|_{\text{LIP}^\infty}$. In this case we obtain that $M^{1,\infty}(X) = \text{LIP}^\infty(X) \subseteq D^\infty(X)$.

(5.2) Newtonian space. Another interesting generalization of Sobolev spaces to general metric spaces are the so-called Newtonian Spaces, introduced by Shanmugalingam [Sh1, Sh2]. Its definition is based on the notion of the upper gradient that we recall here for the sake of completeness.

A non-negative Borel function g on X is said to be an *upper gradient* for an extended real-valued function f on X , if

$$|f(\gamma(a)) - f(\gamma(b))| \leq \int_\gamma g \quad (*)$$

for every rectifiable curve $\gamma : [a, b] \rightarrow X$. We see that the upper gradient plays the role of a derivative in the formula (*) which is similar to the one related to the fundamental theorem of calculus. The point is that using upper gradients we may have many of the properties of ordinary Sobolev spaces even though we do not have derivatives of our functions.

If g is an upper gradient of u and $\tilde{g} = g$ almost everywhere, then it may happen that \tilde{g} is no longer an upper gradient for u . We do not want our upper gradients to be sensitive to changes on small sets. To avoid this unpleasant situation the notion of *weak upper gradient* is introduced as follows. First we need a way to measure how large a family of curves is. The most important point is if a family of curves is small enough to be ignored. This kind of problem was first approached in [Fu]. In what follows let $\Upsilon \equiv \Upsilon(X)$ denote the family of all nonconstant rectifiable curves in X . It may happen $\Upsilon = \emptyset$, but we will be mainly concerned with metric spaces for which the space Υ is large enough.

Definition 5.3. (Modulus of a family of curves) Let $\Gamma \subset \Upsilon$. For $1 \leq p < \infty$ we define the *p-modulus* of Γ by

$$\text{Mod}_p(\Gamma) = \inf_\rho \int_X \rho^p d\mu,$$

where the infimum is taken over all non-negative Borel functions $\rho : X \rightarrow [0, \infty]$ such that $\int_\gamma \rho \geq 1$ for all $\gamma \in \Gamma$. If some property holds for all curves $\gamma \in \Upsilon \setminus \Gamma$, such that $\text{Mod}_p \Gamma = 0$, then we say that the property holds for *p-a.e. curve*.

Definition 5.4. A non-negative Borel function g on X is a *p-weak upper gradient* of an extended real-valued function f on X , if

$$|f(\gamma(a)) - f(\gamma(b))| \leq \int_\gamma g$$

for *p-a.e. curve* $\gamma \in \Upsilon$.

Let $\tilde{N}^{1,p}(X, d, \mu)$, where $1 \leq p < \infty$, be the class of all L^p integrable Borel functions on X for which there exists a p -weak upper gradient in L^p . For $f \in \tilde{N}^{1,p}(X, d, \mu)$ we define

$$\|u\|_{\tilde{N}^{1,p}} = \|u\|_{L^p} + \inf_g \|g\|_{L^p},$$

where the infimum is taken over all p -weak upper gradients g of u . Now, we define in $\tilde{N}^{1,p}$ an equivalence relation by $u \sim v$ if and only if $\|u - v\|_{\tilde{N}^{1,p}} = 0$. Then the space $N^{1,p}(X, d, \mu)$ is defined as the quotient $\tilde{N}^{1,p}(X, d, \mu)/\sim$ and it is equipped with the norm $\|u\|_{N^{1,p}} = \|u\|_{\tilde{N}^{1,p}}$.

Next, we consider the case $p = \infty$. We will introduce the corresponding definition of ∞ -modulus of a family of rectifiable curves which will be an important ingredient for the definition of the Sobolev space $N^{1,\infty}(X)$.

Definition 5.5. For $\Gamma \subset \Upsilon$, let $F(\Gamma)$ be the family of all Borel measurable functions $\rho : X \rightarrow [0, \infty]$ such that

$$\int_{\gamma} \rho \geq 1 \quad \text{for all } \gamma \in \Gamma.$$

We define the ∞ -modulus of Γ by

$$\text{Mod}_{\infty}(\Gamma) = \inf_{\rho \in F(\Gamma)} \{\|\rho\|_{L^{\infty}}\} \in [0, \infty].$$

If some property holds for all curves $\gamma \in \Upsilon \setminus \Gamma$, where $\text{Mod}_{\infty} \Gamma = 0$, then we say that the property holds for ∞ -a.e. curve.

Remark 5.6. It can be easily checked that Mod_{∞} is an outer measure as it happens for $1 \leq p < \infty$. See for example Theorem 5.2 in [Ha1].

Next, we provide a characterization of path families whose ∞ -modulus is zero.

Lemma 5.7. Let $\Gamma \subset \Upsilon$. The following conditions are equivalent:

- (a) $\text{Mod}_{\infty} \Gamma = 0$.
- (b) There exists a Borel function $0 \leq \rho \in L^{\infty}(X)$ such that $\int_{\gamma} \rho = +\infty$, for each $\gamma \in \Gamma$.
- (c) There exists a Borel function $0 \leq \rho \in L^{\infty}(X)$ such that $\int_{\gamma} \rho = +\infty$, for each $\gamma \in \Gamma$ and $\|\rho\|_{L^{\infty}} = 0$.

Proof. (a) \Rightarrow (b) If $\text{Mod}_{\infty} \Gamma = 0$, for each $n \in \mathbb{N}$ there exists $\rho_n \in F(\Gamma)$ such that $\|\rho_n\|_{L^{\infty}} < 1/2^n$. Let $\rho = \sum_{n \geq 1} \rho_n$. Then $\|\rho\|_{L^{\infty}} \leq \sum_{n=1}^{\infty} 1/2^n = 1$ and $\int_{\gamma} \rho = \int_{\gamma} \sum_{n \geq 1} \rho_n = \infty$.

(b) \Rightarrow (a) On the other hand, let $\rho_n = \rho/n$ for all $n \in \mathbb{N}$. By hypothesis $\int_{\gamma} \rho_n = \infty$ for all $n \in \mathbb{N}$ and $\gamma \in \Gamma$. Then $\rho_n \in F(\Gamma)$ and $\|\rho\|_{L^{\infty}}/n \rightarrow 0$ as $n \rightarrow \infty$. Hence $\text{Mod}_{\infty}(\Gamma) = 0$.

(b) \Rightarrow (c) By hypothesis there exists a Borel measurable function $0 \leq \rho \in L^{\infty}(X)$ such that,

$$\int_{\gamma} \rho = +\infty \quad \text{for every } \gamma \in \Gamma.$$

Consider the function

$$h(x) = \begin{cases} \|\rho\|_{L^{\infty}} & \text{if } \|\rho\|_{L^{\infty}} \geq \rho(x), \\ \infty & \text{if } \rho(x) > \|\rho\|_{L^{\infty}}. \end{cases}$$

Notice that $\|\rho\|_{L^\infty} = \|h\|_{L^\infty}$, and since $\int_\gamma \rho = +\infty$ for every $\gamma \in \Gamma_1$ and $\rho \leq h$, we have that $\int_\gamma h = +\infty$ for every $\gamma \in \Gamma_1$. Now, we define the function $\varrho = h - \|h\|_{L^\infty}$ which has $\|\varrho\|_{L^\infty} = 0$ and

$$\int_\gamma \varrho = \int_\gamma h - \|h\|_{L^\infty} \ell(\gamma) = +\infty \quad \text{for every } \gamma \in \Gamma_1.$$

□

Now we are ready to define the notion of ∞ -weak upper gradient.

Definition 5.8. A non-negative Borel function g on X is an ∞ -weak upper gradient of an extended real-valued function f on X , if

$$|f(\gamma(a)) - f(\gamma(b))| \leq \int_\gamma g$$

for ∞ -a.e. curve every curve $\gamma \in \Upsilon$.

Let $\tilde{N}^{1,\infty}(X, d, \mu)$, be the class of all functions $f \in L^\infty(X)$ Borel for which there exists an ∞ -weak upper gradient in L^∞ . For $f \in \tilde{N}^{1,\infty}(X, d, \mu)$ we define

$$\|u\|_{\tilde{N}^{1,\infty}} = \|u\|_{L^\infty} + \inf_g \|g\|_{L^\infty},$$

where the infimum is taken over all ∞ -weak upper gradients g of u .

Definition 5.9. (Newtonian space for $p = \infty$) We define an equivalence relation in $\tilde{N}^{1,\infty}$ by $u \sim v$ if and only if $\|u - v\|_{\tilde{N}^{1,\infty}} = 0$. Then the space $N^{1,\infty}(X, d, \mu)$ is defined as the quotient $\tilde{N}^{1,\infty}(X, d, \mu) / \sim$ and it is equipped with the norm

$$\|u\|_{N^{1,\infty}} = \|u\|_{\tilde{N}^{1,\infty}}.$$

Note that if $u \in \tilde{N}^{1,\infty}$ and $v = u$ μ -a.e., then it is not necessarily true that $v \in \tilde{N}^{1,\infty}$. Indeed, let $(X = [-1, 1], d, \lambda)$ where d denotes the Euclidean distance and λ the Lebesgue measure. Let $u : X \rightarrow \mathbb{R}$ be the function $u = 1$ and $v : X \rightarrow \mathbb{R}$ given by $v = 1$ if $x \neq 0$ and $v(x) = \infty$ if $x = 0$. In this case we have that $u = v$ μ -a.e., $u \in \tilde{N}^{1,\infty}$ but $v \notin \tilde{N}^{1,\infty}$. It can be shown that if $u, v \in \tilde{N}^{1,\infty}$, and $v = u$ μ -a.e., then $\|u - v\|_{\tilde{N}^{1,\infty}} = 0$. In addition, $N^{1,\infty}(X)$ is a Banach space. Both results can be checked adapting properly the respective proofs for the case $p < \infty$. For further details see [Sh2].

Lemma 5.10. If $f \in D(X)$ then $\text{Lip}(f)$ is an upper gradient of f .

Proof. Let $\gamma : [a, b] \rightarrow X$ be a rectifiable curve parametrized by arc-length which connects x and y . It can be checked that γ is 1-Lipschitz (see for instance Theorem 3.2 in [Ha1]). The function $f \circ \gamma$ is an infinitesimally Lipschitz function and by Stepanov's differentiability theorem (see [BRZ]), it is differentiable a.e. Note that $|(f \circ \gamma)'(t)| \leq \text{Lip } f(\gamma(t))$ at every point of $[a, b]$ where $(f \circ \gamma)$ is differentiable. Now, we deduce that

$$|f(x) - f(y)| \leq \left| \int_a^b (f \circ \gamma)'(t) dt \right| \leq \int_a^b \text{Lip}(f(\gamma(t))) dt$$

as wanted. □

Now suppose that $\mu(B) > 0$ for every open ball $B \subset X$. It is clear by Lemma 5.10 that $D^\infty(X) \subset \tilde{N}^{1,\infty}(X)$ and that the map

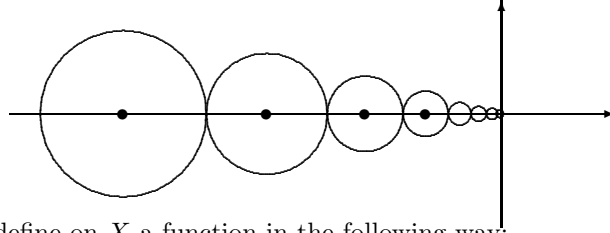
$$\begin{aligned} \phi : D^\infty(X) &\longrightarrow N^{1,\infty}(X) \\ f &\longrightarrow [f]. \end{aligned}$$

is an inclusion. Indeed, if $f, g \in D^\infty(X)$ with $0 = [f - g] \in N^{1,\infty}(X)$, we have $f - g = 0$ μ -a.e. Thus $f = g$ in a dense subset and since f, g are continuous we obtain that $f = g$. Therefore we have the following chain of inclusions:

$$\text{LIP}^\infty(X) = M^{1,\infty}(X) \subset D^\infty(X) \subset N^{1,\infty}(X), \quad (*)$$

and $\|\cdot\|_{N^{1,\infty}} \leq \|\cdot\|_{D^\infty} \leq \|\cdot\|_{\text{LIP}^\infty} \leq 2\|\cdot\|_{M^{1,\infty}}$. The next example shows that in general $D^\infty(X) \neq N^{1,\infty}(X)$.

Example 5.11. Consider the metric space $(X = \{B_n\}_n, d_e)$, where d_e is the restriction to X of the Euclidean metric of \mathbb{R}^2 and $\{B_n\}_n$ is a sequence of open balls with radius convergent to zero, as shows the picture below:



We define on X a function in the following way:

$$f(x, y) = \begin{cases} 1 & \text{if } (x, y) \in B_i \quad i = 2k + 1 \quad k \in \mathbb{Z}, \\ 0 & \text{if } (x, y) \in B_i \quad i = 2k \quad k \in \mathbb{Z}. \end{cases}$$

The constant function $g = 0$ is clearly an upper gradient of f , and so $f \in N^{1,\infty}(X)$. But, there is no continuous representative for the function f . Thus, in particular, f does not admit a representative in D^∞ .

In the following, we will look for conditions under which the Sobolev spaces $M^{1,\infty}(X)$ and $N^{1,\infty}(X)$ coincide. In particular, this will give us the equality of all the spaces in the chain $(*)$ above. For that, we need some preliminary terminology and results.

Definition 5.12. We say that a measure μ on X is *doubling* if there is a positive constant C_μ such that

$$0 < \mu(B(x, 2r)) \leq C_\mu \mu(B(x, r)) < \infty,$$

for each $x \in X$ and $r > 0$. Here $B(x, r)$ denotes the open ball of center x and radius $r > 0$.

Definition 5.13. Let $1 \leq p < \infty$. We say that (X, d, μ) supports a *weak p -Poincaré inequality* if there exist constants $C_p > 0$ and $\lambda \geq 1$ such that for every Borel measurable function $u : X \rightarrow \mathbb{R}$ and every upper gradient $g : X \rightarrow [0, \infty]$ of u , the pair (u, g) satisfies the inequality

$$\int_{B(x, r)} |u - u_{B(x, r)}| d\mu \leq C_p r \left(\int_{B(x, \lambda r)} g^p d\mu \right)^{1/p}$$

for each $B(x, r) \subset X$.

Here for arbitrary $A \subset X$ with $0 < \mu(A) < \infty$ we write

$$\int_A f = \frac{1}{\mu(A)} \int_A f d\mu.$$

The Poincaré inequality creates a link between the measure, the metric and the gradient and it provides a way to pass from the infinitesimal information which gives the gradient to larger scales. Metric spaces with doubling measure and Poincaré

inequality admit first order differential calculus akin to that in Euclidean spaces. See [Am], [He1] or [He2] for further information about these topics.

The proof of the next result is strongly inspired in Proposition 3.2 in [JJRRS]. However, we include all the details because of the technical differences, which at certain points become quite subtle.

Theorem 5.14. *Let X be a complete metric space that supports a doubling Borel measure μ which is non-trivial and finite on balls and suppose that X supports a weak p -Poincaré inequality for some $1 \leq p < \infty$. Let $\rho \in L^\infty(X)$ such that $0 \leq \rho$. Then, there exists a set $F \subset X$ of measure 0 and a constant $K > 0$ (depending only on X) such that for all $x, y \in X \setminus F$ there exist a rectifiable curve γ such that $\int_\gamma \rho < +\infty$ and $\ell(\gamma) \leq Kd(x, y)$.*

Proof. We may assume that $0 < \|\rho\|_{L^\infty} \leq 1$. Indeed, in other case, we could take $\tilde{\rho} = \rho/(1 + \|\rho\|_{L^\infty})$. Let $E = \{x \in X : \rho(x) > \|\rho\|_{L^\infty}\}$, which is a set of measure zero. By Theorem 2.2 in [He1], there exists a constant C depending only on the doubling constant C_μ of X such that for each $f \in L^1(X)$ and for all $t > 0$

$$\mu(\{M(f) > t\}) \leq \frac{C}{t} \int_X |f| d\mu$$

Recall that $M(f)(x) = \sup_{r>0} \{ \int_{B(x,r)} |f| d\mu \}$.

For each $n \geq 1$ we can choose V_n be an open set such that $E \subset V_n$ and $\mu(V_n) \leq (\frac{1}{n2^n})^p$ (see Theorem 1.10 in [M]). Note that $E \subseteq \bigcap_{n \geq 1} V_n = E_0$ and $\mu(E_0) = \mu(E) = 0$.

Next, consider the family of functions

$$\rho_n = \|\rho\|_{L^\infty} + \sum_{m \geq n} \chi_{V_m}$$

and the function ρ_0 given by the formula

$$\rho_0(x) = \begin{cases} \|\rho\|_{L^\infty} & \text{if } x \in X \setminus E_0, \\ +\infty & \text{otherwise.} \end{cases}$$

We have the following properties:

- (i) $\rho_n|_{X \setminus V_n} \equiv \|\rho\|_{L^\infty}$.
- (ii) $\rho \leq \rho_0 \leq \rho_m \leq \rho_n$ if $n \leq m$.
- (iii) $\rho_n|_{E_0} \equiv +\infty$.
- (iv) $\rho_n \in L^p(X)$ is lower semicontinuous; in fact $\|\rho_n - \|\rho\|_{L^\infty}\|_{L^p} \leq \frac{1}{n}$.

Indeed, since each of the sets V_m are open then the functions χ_{V_m} are lower semicontinuous (see Proposition 7.11 in [F]) and so once we check that $\|\rho_n - \|\rho\|_{L^\infty}\|_{L^p} \leq \frac{1}{n}$, we will be done. For that, it is enough to prove that $\sum_{m \geq n} \|\chi_{V_m}\|_{L^p} \leq \frac{1}{n}$, which follows from the formula

$$\sum_{m \geq n} \|\chi_{V_m}\|_{L^p} = \sum_{m \geq n} (\mu(V_m))^{1/p} = \sum_{m \geq n} \frac{1}{m2^m} \leq \frac{1}{n} \sum_{m \geq n} \frac{1}{2^m} \leq \frac{1}{n}.$$

- (v) $\mu(\{M((\rho_n - \|\rho\|_{L^\infty})^p) > 1\}) \leq \frac{C}{n^p}$.

Indeed, as we have seen above

$$\begin{aligned} \mu(\{M((\rho_n - \|\rho\|_{L^\infty})^p) > 1\}) &\leq \frac{C}{1} \int_X |\rho_n - \|\rho\|_{L^\infty}|^p \\ &= C \|\rho_n - \|\rho\|_{L^\infty}\|_{L^p}^p < C \frac{1}{n^p}. \end{aligned}$$

For each $n \geq 1$ consider the set

$$S_n = \{x \in X : M((\rho_n - \|\rho\|_{L^\infty})^p)(x) \leq 1\}$$

We claim that: $S_n \subset S_m$ if $n \leq m$ and $F = X \setminus \bigcup_{n \geq 1} S_n$ has measure 0.

Indeed, if $n \leq m$, we have that $0 \leq \rho_m - \|\rho\|_{L^\infty} \leq \rho_n - \|\rho\|_{L^\infty}$ and so

$$0 \leq (\rho_m - \|\rho\|_{L^\infty})^p \leq (\rho_n - \|\rho\|_{L^\infty})^p;$$

hence $S_n \subset S_m$. On the other hand by (v) above, we have $\mu(X \setminus S_n) \leq \frac{C}{n^p}$. Thus,

$$0 \leq \mu(F) = \mu\left(X \setminus \bigcup_{n \geq 1} S_n\right) = \mu\left(\bigcap_{n \geq 1} (X \setminus S_n)\right) = \lim_{n \rightarrow \infty} \mu(X \setminus S_n) \leq \lim_{n \rightarrow \infty} \frac{C}{n^p} = 0.$$

After all this preparatory work, our aim is to prove that there exists a constant $K > 0$ depending only on X such that for all $x, y \in X \setminus F$ there exist a rectifiable curve γ such that $\int_\gamma \rho < +\infty$ and $\ell(\gamma) \leq Kd(x, y)$. The constant K will be constructed along the remainder of the proof. In what follows let m_0 be the smallest integer for which $S_{m_0} \neq \emptyset$. Fix $n \geq m_0$ and a point $x_0 \in S_n \subset X \setminus F$. As one can check straightforwardly, it is enough to prove that for each $x \in S_n$ there exists a rectifiable curve γ such that $\int_\gamma \rho < +\infty$ and $\ell(\gamma) \leq Kd(x, y)$, where the constant K depends only on X and not on x_0 or n .

For our purposes, we define the set Γ_{xy} as the set of all the rectifiable curves connecting x and y . Since a complete metric space X supporting a doubling measure and a weak p -Poincaré inequality is quasi-convex (see Theorem 17.1 in [Ch]), it is clear that Γ_{xy} is nonempty. We define the function

$$u_n(x) = \inf \left\{ \ell(\gamma) + \int_\gamma \rho_n : \gamma \in \Gamma_{x_0 x} \right\}.$$

Note that $u_n(x_0) = 0$. We will prove that in S_n the function u_n is bounded by a Lipschitz function v_n with a constant K_0 which depends only on X and $\|\rho\|_{L^\infty}$ (and not on x_0 nor n) such that $v_n(x_0) = 0$. Assume this for a moment. We have

$$0 \leq u_n(x) = u_n(x) - u_n(x_0) \leq v_n(x) - v_n(x_0) \leq K_0 d(x, x_0) < (K_0 + 1)d(x, x_0).$$

Thus, there exists a rectifiable curve $\gamma \in \Gamma_{x_0 x}$ such that

$$\ell(\gamma) + \int_\gamma \rho \leq \ell(\gamma) + \int_\gamma \rho_n \leq (K_0 + 1)d(x, x_0).$$

Hence, taking $K = K_0 + 1$, we will have

$$\ell(\gamma) \leq Kd(x, x_0) \quad \text{and} \quad \int_\gamma \rho < +\infty,$$

as we wanted.

Therefore, consider the functions $u_{n,k} : X \rightarrow \mathbb{R}$ given by

$$u_{n,k} = \inf \left\{ \ell(\gamma) + \int_\gamma \rho_{n,k} : \gamma \in \Gamma_{x_0 x} \right\}$$

where $\rho_{n,k} = \min\{\rho_n, k\}$ which is a lower semicontinuous function. Let us see that the functions $u_{n,k}$ are Lipschitz for each $k \geq 1$ (and in particular continuous) and that $\rho_{n,k} + 1 \leq \rho_n + 1$ are upper gradients for $u_{n,k}$. Since X is quasi-convex, it follows that $u_{n,k}(x) < +\infty$ for all $x \in X$.

Indeed, let $y, z \in X$, C_q the constant of quasi-convexity for X and $\varepsilon > 0$. We may assume that $u_{n,k}(z) \geq u_{n,k}(y)$. Let $\gamma_y \in \Gamma_{x_0 y}$ be such that

$$u_{n,k}(y) \geq \ell(\gamma_y) + \int_{\gamma_y} \rho_{n,k} - \varepsilon.$$

On the other hand, for each rectifiable curve $\gamma_{yz} \in \Gamma_{yz}$, we have

$$u_{n,k}(z) \leq \ell(\gamma_y \cup \gamma_{yz}) + \int_{\gamma_y \cup \gamma_{yz}} \rho_{n,k},$$

and so

$$|u_{n,k}(z) - u_{n,k}(y)| = u_{n,k}(z) - u_{n,k}(y) \leq \ell(\gamma_{yz}) + \int_{\gamma_{yz}} \rho_{n,k} = \int_{\gamma_{yz}} (\rho_{n,k} + 1).$$

Thus, $\rho_{n,k} + 1$ is an upper gradient for $u_{n,k}$. In particular, if $\ell(\gamma_{zy}) \leq C_q d(z, y)$, we deduce that

$$|u_{n,k}(z) - u_{n,k}(y)| \leq (k+1)\ell(\gamma_{zy}) \leq C_q(k+1)d(z, y)$$

and so $u_{n,k}$ is a $C_q(k+1)$ -Lipschitz function. Our purpose now is to prove that the restriction to S_n of each function $u_{n,k}$ is a Lipschitz function on S_n with respect to a constant K_0 which depends only on X . Fix $y, z \in S_n$. For each $i \in \mathbb{Z}$, define $B_i = B(z, 2^{-i}, d(z, y))$ if $i \geq 1$, $B_0 = B(z, 2d(z, y))$, and $B_i = B(y, 2^i d(z, y))$ if $i \leq -1$. To simplify notation we write $\lambda B(x, r) = B(x, \lambda r)$. In the first inequality of the following estimation we use the fact that, since $u_{n,k}$ is continuous, all points of X are Lebesgue points of $u_{n,k}$. Using the weak p -Poincaré inequality and the doubling condition we get the third inequality. From the Minkowski inequality we deduce the fifth while the last one follows from the definition of S_n :

$$\begin{aligned} |u_{n,k}(z) - u_{n,k}(y)| &\leq \sum_{i \in \mathbb{Z}} \left| \int_{B_i} u_{n,k} d\mu - \int_{B_{i+1}} u_{n,k} d\mu \right| \\ &\stackrel{(*)}{\leq} \sum_{i \in \mathbb{Z}} \frac{1}{\mu(B_i)} \int_{B_i} |u_{n,k} - \int_{B_{i+1}} u_{n,k} d\mu| d\mu \\ &\leq C_\mu C_p d(z, y) \sum_{i \in \mathbb{Z}} 2^{-|i|} \left(\frac{1}{\mu(\lambda B_i)} \int_{\lambda B_i} (\rho_{n,k} + 1)^p \right)^{1/p} \\ &\leq C_\mu C_p d(z, y) \sum_{i \in \mathbb{Z}} 2^{-|i|} \left(\frac{1}{\mu(\lambda B_i)} \int_{\lambda B_i} ((\rho_{n,k} - \|\rho\|_{L^\infty}) + \|\rho\|_{L^\infty} + 1)^p \right)^{1/p} \\ &\leq C_\mu C_p d(z, y) \sum_{i \in \mathbb{Z}} 2^{-|i|} \left(\|\rho\|_{L^\infty} + 1 + \left(\frac{1}{\mu(\lambda B_i)} \int_{\lambda B_i} (\rho_{n,k} - \|\rho\|_{L^\infty})^p \right)^{1/p} \right) \\ &\leq 3 C_\mu C_p d(z, y) \sum_{i \in \mathbb{Z}} 2^{-|i|} \leq K_0 d(z, y) \end{aligned}$$

where $K_0 = 9C_\mu C_p$ is a constant that depends only on X . Recall that C_μ is the doubling constant and C_p is the constant which appears in the weak p -Poincaré inequality. Let us see with more detail inequality (*). If $i > 0$, we have that

$$\begin{aligned} \left| \int_{B_i} u_{n,k} d\mu - \int_{B_{i+1}} u_{n,k} d\mu \right| &\leq \frac{1}{\mu(B_{i+1})} \left| \int_{B_{i+1}} (u_{n,k} - \int_{B_i} u_{n,k} d\mu) d\mu \right| \\ &\leq \frac{\mu(B_i)}{\mu(B_i) \mu(B_{i+1})} \left| \int_{B_i} (u_{n,k} - \int_{B_i} u_{n,k} d\mu) d\mu \right| \\ &\leq \frac{C_\mu}{\mu(B_i)} \left| \int_{B_i} (u_{n,k} - \int_{B_i} u_{n,k} d\mu) d\mu \right|. \end{aligned}$$

We have used that $B_{i+1} \subset B_i$ for $i > 0$ and that μ is a doubling measure and so $\mu(2B_{i+1}) = \mu(B_i) \leq C_\mu \mu(B_{i+1})$. The cases $i < 0$ and $i = 0$ are similar.

Thus, the restriction of $u_{n,k}$ to S_n is a K_0 -Lipschitz function for all $k \geq 1$. Note that $u_{n,k} \leq u_{n,k+1}$ and therefore we may define

$$v_n(x) = \sup_k \{u_{n,k}(x)\} = \lim_{k \rightarrow \infty} u_{n,k}(x).$$

Whence v_n is a K_0 -Lipschitz function on S_n . Since $v(x_0) = 0$ and $x_0 \in S_m$ when $m \geq m_0$ we have that $v(x) < \infty$ and so, it is enough to check that $u_n(x) \leq v_n(x)$ for $x \in S_n$. Now, fix $x \in S_n$. For each $k \geq 1$ there is $\gamma_k \in \Gamma_{x_0 x}$ such that

$$\ell(\gamma_k) + \int_{\gamma_k} \rho_{n,k} \leq u_{n,k}(x) + \frac{1}{k} \leq K_0 d(x, x_0) + \frac{1}{k}.$$

In particular, $\ell(\gamma_k) \leq K_0 d(x, x_0) + 1 := M$ for every $k \geq 1$ and so, by reparametrization, we may assume that γ_k is an M -Lipschitz function and $\gamma_k : [0, 1] \rightarrow \overline{B(x_0, M)}$ for all $k \geq 1$. Since X is complete and doubling, and therefore closed balls are compact, we are in a position to use the Ascoli-Arzelà theorem to obtain a subsequence $\{\gamma_k\}_k$ (which we denote again by $\{\gamma_k\}_k$ to simplify notation) and $\gamma : [0, 1] \rightarrow X$ such that $\gamma_k \rightarrow \gamma$ uniformly. For each k_0 , the function $1 + \rho_{n,k_0}$ is lower semicontinuous, and therefore by Lemma 2.2 in [JJRRS] and the fact that $\{\rho_{n,k}\}$ is an increasing sequence of functions, we have

$$\ell(\gamma) + \int_{\gamma} \rho_{n,k_0} = \int_{\gamma} (1 + \rho_{n,k_0}) \leq \liminf_{k \rightarrow \infty} \int_{\gamma_k} (1 + \rho_{n,k_0}) \leq \liminf_{k \rightarrow \infty} \int_{\gamma_k} (1 + \rho_{n,k}).$$

Using the monotone convergence theorem on the left hand side and letting k_0 tend to infinity yields

$$\ell(\gamma) + \int_{\gamma} \rho_n \leq \liminf_{k \rightarrow \infty} \int_{\gamma_k} (1 + \rho_{n,k}).$$

Since $\gamma \in \Gamma_{x_0 x}$ we have

$$\begin{aligned} u_n(x) &\leq \ell(\gamma) + \int_{\gamma} \rho_n \leq \liminf_{k \rightarrow \infty} \int_{\gamma_k} (1 + \rho_{n,k}) \\ &\leq \liminf_{k \rightarrow \infty} \left(u_{n,k}(x) + \frac{1}{k} \right) \leq v_n(x), \end{aligned}$$

and that completes the proof. \square

Remark 5.15. In Theorem 5.14 we can change the hypothesis of completeness for the space X by local compactness. The proof is analogous to the one of Theorem 1.6 in [JJRRS], and we do not include the details.

Corollary 5.16. *Let X be a complete metric space that supports a doubling Borel measure μ which is non-trivial and finite on balls. If X supports a weak p -Poincaré inequality for $1 \leq p < \infty$, then $\text{LIP}^\infty(X) = M^{1,\infty}(X) = N^{1,\infty}(X)$ with equivalent norms.*

Proof. If $f \in N^{1,\infty}(X)$, then there exists an ∞ -weak upper gradient $g \in L^\infty(X)$ of f . We denote Γ_1 the family of curves for which g is not an upper gradient for f . Note that $\text{Mod}_\infty \Gamma_1 = 0$. By Lemma 5.7 there exists a Borel measurable function $0 \leq \varrho \in L^\infty(X)$ such that, $\int_{\gamma} \varrho = +\infty$ for every $\gamma \in \Gamma_1$ and $\|\varrho\|_{L^\infty} = 0$. Consider $\rho_0 = g + \varrho \in L^\infty(X)$ which is an upper gradient of f and satisfies that $\|\rho_0\|_{L^\infty} = \|g\|_{L^\infty}$. Note that $\int_{\gamma} \rho_0 = +\infty$ for all $\gamma \in \Gamma_1$ and that by Lemma 5.7 the family of curves $\Gamma_2 = \{\gamma \in \Upsilon : \int_{\gamma} \rho_0 = +\infty\}$ has ∞ -modulus zero. Finally, consider the set $\{x \in X : g(x) + \varrho(x) \geq \|\rho_0\|_{L^\infty}\}$ and define

$$\rho(x) = \begin{cases} \|\rho_0\|_{L^\infty} & \text{if } x \in X \setminus E, \\ +\infty & \text{if } x \in E. \end{cases}$$

Then ρ is an upper gradient of f and it satisfies that $\|\rho\|_{L^\infty} = \|\rho_0\|_{L^\infty} = \|g\|_{L^\infty}$. Note that if $\int_{\gamma} \rho < +\infty$, then the set $\gamma^{-1}(+\infty)$ has measure zero in the domain of γ (because otherwise $\int_{\gamma} \rho = +\infty$). Thus, if $\int_{\gamma} \rho < +\infty$, we have in particular that $\int_{\gamma} \rho = \|\rho\|_{L^\infty} \ell(\gamma)$. By Theorem 5.14 there exists a set $F \subset X$ of measure 0 and a

constant $K > 0$ (depending only on X) such that for all $x, y \in X \setminus F$ there exist a rectifiable curve γ such that $\int_\gamma \rho < +\infty$ and $\ell(\gamma) \leq Kd(x, y)$. Let now $x, y \in X \setminus F$ and γ be a rectifiable curve satisfying the precedent conditions. Then

$$|f(x) - f(y)| \leq \int_\gamma \rho \stackrel{(*)}{=} \|\rho\|_{L^\infty} \ell(\gamma) \leq \|\rho\|_{L^\infty} Kd(x, y).$$

Then f is $\|\rho\|_{L^\infty} K$ -Lipschitz a.e. Thus, $\text{LIP}^\infty(X) = M^{1,\infty}(X) = N^{1,\infty}(X)$. \square

Remark 5.17. Note that if we would have chosen as upper gradient ρ_0 instead of ρ , the inequality $(*)$ might not be necessary true. To see this, it is enough to define a function which is zero a.e. and constant but finite on a set of zero measure.

Our purpose now is to see under which conditions the spaces $D^\infty(X)$ and $N^{1,\infty}(X)$ coincide. For that, we need first to use the local version of the weak p -Poincaré inequality (see for example Definition 4.2.17 in [Sh1]).

Definition 5.18. Let $1 \leq p < \infty$. We say that (X, d, μ) supports a *local weak p -Poincaré inequality* with constant C_p if for every $x \in X$, there exists a neighborhood U_x of x and $\lambda \geq 1$ such that whenever B is a ball in X such that λB is contained in U_x , and u is an integrable function on λB with g as its upper gradient in λB , then

$$\int_{B(x,r)} |u - u_{B(x,r)}| d\mu \leq C_p r \left(\int_{B(x,\lambda r)} g^p d\mu \right)^{1/p}.$$

Corollary 5.19. Let X be a complete metric space that supports a doubling Borel measure μ which is non-trivial and finite on balls. If X supports a local weak p -Poincaré inequality for $1 \leq p < \infty$. Then $N^{1,\infty}(X) = D^\infty(X)$ with equivalent norms.

Proof. If $f \in N^{1,\infty}(X)$, then there exists an ∞ -weak upper gradient $g \in L^\infty(X)$ of f . We construct in the same way as in Corollary 5.16 an upper gradient ρ of f which satisfies $\|\rho\|_{L^\infty} = \|g\|_{L^\infty}$, $\int_\gamma \rho \geq |f(\gamma(0)) - f(\gamma(L))|$ for all $\gamma \in \Upsilon$ and $\int_\gamma \rho = \|\rho\|_{L^\infty} \ell(\gamma)$ for all $\gamma \in \Upsilon$ such that $\int_\gamma \rho < +\infty$. Fix $x \in X$. Using a local version of Theorem 5.14 we obtain that there exists a neighborhood U^x and a constant $K > 0$ (depending only on X) such that for almost every $z, y \in U^x$, there exist a rectifiable curve γ connecting z and y such that $\int_\gamma \rho < +\infty$ and $\ell(\gamma) \leq Kd(z, y)$. Let now $y \in U^x$ and γ a rectifiable curve satisfying the precedent conditions. Then

$$|f(x) - f(y)| \leq \int_\gamma \rho = \|\rho\|_{L^\infty} \ell(\gamma) \leq \|\rho\|_{L^\infty} Kd(x, y).$$

Under the hypothesis of the corollary it can be easily checked that f is continuous on X and so, there is no obstruction to take the superior limit

$$\limsup_{y \rightarrow x} \frac{|f(x) - f(y)|}{d(x, y)}.$$

Thus, we deduce that $\text{Lip } f(x) \leq K\|\rho\|_{L^\infty}$. Since this is true for each $x \in X$, we have $\|\text{Lip } f\|_\infty \leq K\|\rho\|_{L^\infty} < +\infty$ and we conclude that $f \in D^\infty(X)$. \square

Observe that under the hypothesis of Corollary 5.19 we have that X is a locally radially quasiconvex metric space. We see throughout a very simple example that in general there exist metric spaces X for which the following holds:

$$\text{LIP}^\infty(X) = M^{1,\infty}(X) \subsetneq D^\infty(X) = N^{1,\infty}(X).$$

Indeed, consider the metric space (X, d, λ) where $X = \mathbb{C} \setminus \{\operatorname{Re}(z) \geq 0, |\operatorname{Im}(z)| \leq 1/2\}$, d is the metric induced by the Euclidean one and λ denotes the Lebesgue measure. Since X is a complete metric space that supports a doubling measure and a local weak p -Poincaré inequality for any $1 \leq p < \infty$, by Corollary 5.19, we have that $D^\infty(X) = N^{1,\infty}(X)$. Let $f(z) = \arg(z)$, for each $z \in X$. One can check that $f \in D^\infty(X) = N^{1,\infty}(X)$. However, $f \notin \operatorname{LIP}^\infty(X)$, and so $\operatorname{LIP}^\infty(X) \subsetneq D^\infty(X) = N^{1,\infty}(X)$.

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